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STABILITY AND ERGODICITY OF PIECEWISE DETERMINISTIC MARKOV PROCESSES

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Abstract. The main goal of this paper is to establish some equivalence results on stability, recurrence, and ergodicity between a piecewise deterministic Markov process (PDMP) \( \{X(t)\} \) and an embedded discrete-time Markov chain \( \{\Theta_n\} \) generated by a Markov kernel \( G \) that can be explicitly characterized in terms of the three local characteristics of the PDMP, leading to tractable criterion results. First we establish some important results characterizing \( \{\Theta_n\} \) as a sampling of the PDMP \( \{X(t)\} \) and deriving a connection between the probability of the first return time to a set for the discrete-time Markov chains generated by \( G \) and the resolvent kernel \( R \) of the PDMP. From these results we obtain equivalence results regarding irreducibility, existence of \( \sigma \)-finite invariant measures, and (positive) recurrence and (positive) Harris recurrence between \( \{X(t)\} \) and \( \{\Theta_n\} \), generalizing the results of [F. Dufour and O. L. V. Costa, SIAM J. Control Optim., 37 (1999), pp. 1483–1502] in several directions. Sufficient conditions in terms of a modified Foster–Lyapunov criterion are also presented to ensure positive Harris recurrence and ergodicity of the PDMP. We illustrate the use of these conditions by showing the ergodicity of a capacity expansion model.

Key words. piecewise deterministic Markov process, recurrence, ergodicity

AMS subject classifications. 60J25, 60J10, 93E15

DOI. 10.1137/060670109

1. Introduction. Piecewise deterministic Markov processes (PDMPs) were introduced in the literature by Davis [6] as a general class of stochastic models. PDMPs are a family of Markov processes involving deterministic motion punctuated by random jumps. The motion of the PDMP \( \{X(t)\} \) depends on three local characteristics, namely, the flow \( \Phi \), the jump rate \( \lambda \), and the transition measure \( Q \), which specifies the postjump location. Starting from \( x \) the motion of the process follows the flow \( \Phi(x,t) \) until the first jump time \( T_1 \), which occurs either spontaneously in a Poisson-like fashion with rate \( \lambda \) or when the flow \( \Phi(x,t) \) hits the boundary of the state space. In either case the location of the process at the jump time \( T_1 \) is selected by the transition measure \( Q(\Phi(x,T_1),.) \) and the motion restarts from this new point as before. A suitable choice of the state space and the local characteristics \( \Phi \), \( \lambda \), and \( Q \) provide stochastic models covering a great number of problems in operations research [6].

Over the past decades a great deal of attention has been given to the stability properties and related ergodic theory of Markov processes. One of the main approaches to dealing with these problems is to show that the recurrence properties of the Markov process under consideration are related to the recurrence properties of an associated discrete-time Markov chain obtained from a sampling of the original process, so that the well-known discrete-time Markov chains results could be used (see, for example, the books [12, 17, 18] and the references therein).
In the continuous-time context, Azéma, Duflo, and Revuz [1, 2] showed that a general Markov process and its associated resolvent admit the same recurrence properties. It was proved by Tuominen and Tweedie [19] that the recurrence structure of a Markov process \(\{X(t)\}\) with transition semigroup \(\{P^t\}\) and the Markov chain with kernel \(K_F = \int P^t F(dt)\), where \(F\) is a distribution on \([0, \infty)\), are essentially equivalent, provided that a continuity assumption on \(\{P^t\}\) is satisfied, an assumption later suppressed in a fundamental paper by Meyn and Tweedie [11]. It must be pointed out that these results are related to the sampling of a continuous-time process \(\{X(t)\}\), sampled at random times defined by an independent undelayed renewal process. This idea of randomized sampling was generalized to state-dependent sampling to provide some more powerful state-dependent drift criteria in order to ensure stability of the original Markov process. Within this context, Malyšev and Men’sikov [9] derived a modified Foster–Lyapunov criterion to establish recurrence properties for discrete-time Markov chains with countable state space. Meyn and Tweedie [15] generalized this work to discrete-time Markov chains with a general state space and furthermore obtained state-dependent drift conditions to get geometric ergodic properties. The generalization to continuous-time models was established by Dai and Meyn [5] in the context of general state space Markovian queuing models. In particular, they provided sufficient conditions for the existence of bounds on the long-run average moments and rates of convergence of the \(p\)th moments to their steady-state values. Another paper related to this subject is [4].

The main goal of this paper is to establish equivalence results on stability, recurrence, and ergodicity between a PDMP and a discrete-time Markov chain generated by a kernel \(G\) (see (2.2)–(2.4) for its definition) that can be explicitly characterized in terms of the three local characteristics of the PDMP leading to tractable criterion results. From a practical point of view, it should be noted that the results developed in [2, 11, 19] would be hard to apply for PDMPs because the transition semigroup of the PDMP as well as its associated resolvent kernel cannot be explicitly calculated from its local characteristics, which is not the case regarding the kernel \(G\). As shown in Theorem 3.1 below, \(G\) generates a Markov chain that corresponds to a state-dependent sampling of the PDMP \(\{X(t)\}\), providing an interesting parallel between our work in the continuous-time context and the results obtained in [15] in the discrete-time setting. However, it must be stressed that [15] provides general sufficient conditions to ensure that stability of the sampled chain implies stability of the Markov process, but not the converse. One of the main goals of our paper is to show the converse for PDMPs and, in fact, that the PDMP and the discrete-time Markov chain generated by this tractable kernel \(G\) have an equivalent recurrence structure. This is an important feature that distinguishes our work from [15]. We show that the following equivalence results hold:

(i) The PDMP \(\{X(t)\}\) is irreducible if and only if the Markov chain \(\{\Theta_n\}\) associated to \(G\) is irreducible; see Proposition 4.1.

(ii) There is a one-to-one correspondence between the set of invariant measures for the PDMP \(\{X(t)\}\) and for the Markov chain \(\{\Theta_n\}\) associated to \(G\); see Theorem 4.2.

(iii) The PDMP \(\{X(t)\}\) is recurrent if and only if the Markov chain \(\{\Theta_n\}\) associated to \(G\) is recurrent; see Theorem 4.4.

(iv) The PDMP is Harris recurrent if and only if the Markov chain associated to \(G\) is Harris recurrent; see Theorem 4.6.

(v) The PDMP is positive recurrent (respectively, positive Harris recurrent) if and
It is interesting to note that (v) also highlights some differences between our approach and some general results in the literature [2, 11, 15, 19]. Indeed, as shown in [11, Theorem 3.1], the Markov chain generated by \( K_F \) is positive Harris recurrent if and only if the process \( \{X(t)\} \) is positive Harris recurrent, while in our case the PDMP is positive Harris recurrent if and only if the Markov chain associated to \( G \) is Harris recurrent with its unique invariant measure satisfying a boundedness condition, which is far less demanding than positive Harris recurrence.

It should be pointed out that the proof of the Harris recurrence equivalence (item (iv)) requires two important results. One of them, Theorem 3.3, establishes a connection between the probability of the first return time to a set considering the Markov kernels generated by \( G \) and the resolvent kernel \( R \). From this result we obtain the first part of the equivalence; if the process \( \{X(t)\} \) is Harris recurrent, then so is the Markov chain \( \{\Theta_n\} \). The other result is presented in Theorem 3.1, which provides the sample path characterization of the Markov chain generated by \( G \) through the PDMP \( \{X(t)\} \). From this result we get the second part of the equivalence; if the Markov chain \( \{\Theta_n\} \) is Harris recurrent, then so is the PDMP \( \{X(t)\} \).

After we obtain the recurrence structure for PDMPs in terms of \( G \), a natural question that would arise is what could be said about tractable ergodicity conditions for the PDMPs. As shown in [13, Theorem 6.1], if \( \{X(t)\} \) is positive Harris recurrent, then it is ergodic if and only if some skeleton chain is irreducible. Again, the problem is that the transition semigroup and consequently the skeleton chain for \( \{X(t)\} \) cannot be explicitly calculated from the local characteristics of the PDMP. We provide a tractable and equivalent condition to ensure that a skeleton chain is irreducible.

We conclude the paper by presenting sufficient conditions in a modified Foster–Lyapunov criterion form, written in terms of \( G \), to ensure the following for the PDMP: the existence of an invariant probability measure, positive Harris recurrence, and ergodicity.

The paper is organized as follows. In section 2 we present some basic definitions related to the motion of a PDMP, introduce the Markov kernel \( G \), and recall some classical definitions related to Markov processes both in the discrete-time and continuous-time contexts. Some preliminary results are derived in section 3 that will be important to obtain the equivalence properties for the stability of the PDMPs and the Markov kernel \( G \). In section 4, it will be established that the stability and ergodic properties are equivalent for the PDMPs and the kernel \( G \). In section 5 we establish some sufficient conditions to ensure various concepts of stability of the PDMP through a Foster–Lyapunov criterion for the kernel \( G \). Section 6 illustrates the use of these conditions by showing the ergodicity of a capacity expansion model. The last two sections (7 and 8) are devoted to the proofs of Theorems 3.1 and 3.3.

2. Definition of the PDMP and the Markov kernel \( G \). In this section we first present some standard notation and some basic definitions related to the motion of a PDMP \( \{X(t)\} \). For further details the reader is referred to [6]. Afterwards we introduce the Markov kernel \( G \), which we will use for characterizing the recurrence and the Harris recurrence structure of the PDMP \( \{X(t)\} \). At the end of this section, we recall some classical definitions related to Markov processes both in the discrete-time and continuous-time contexts. For a complete exposition on the subject, the reader is referred to the works of Meyn and Tweedie [12, 10, 14, 13]. We follow closely the
notation in Meyn and Tweedie [12].

Let $\mathbb{R}$ be the set of nonnegative real numbers. The set of natural numbers is denoted by $\mathbb{N}$, and $\mathbb{N}^* = \mathbb{N} - \{0\}$. For any metric space $H$, the Borel $\sigma$-field of $H$ is denoted by $\mathcal{B}(H)$. The indicator of a set $A$ is denoted by $1_A$ ($1_A(x) = 1$ if $x \in A$, $1_A(x) = 0$ if $x \notin A$). Let $E$ and $F$ be two metric spaces. A kernel $K$ on $E \times \mathcal{B}(F)$ is a map $K : E \times \mathcal{B}(F) \rightarrow \mathbb{R} \cup \{+\infty\}$ such that for $x \in E$, $K(x,.)$ is a nonnegative $\sigma$-finite measure on $(F, \mathcal{B}(F))$ and for any $A \in \mathcal{B}(F)$, $K(.,A)$ is a measurable function on $E$. The kernel $I_A$ on $(E, \mathcal{B}(E))$ is defined for any set $A \in \mathcal{B}(E)$ by $I_A(x,B) = 1_{A \cap B}(x)$. For two kernels $K_1$ and $K_2$ defined on $E \times \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+)$, the convolution $K_1 * K_2$ is again a kernel on $E \times \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+)$ defined as

$$K_1 * K_2(x, A \times \Gamma) \doteq \int_E \int_{\mathbb{R}_+} K_1(x, dy \times dt) K_2(y, A \times (\Gamma - t)).$$

where $A \in \mathcal{B}(E)$, $\Gamma \in \mathcal{B}(\mathbb{R}_+)$, and $\Gamma - t = \{u - t : u \in \Gamma\}$. The notation $K^n$ represents the $n$-fold convolution $K \ast \cdots \ast K$. For a positive real-valued measurable function $f$ defined on $E \times \mathbb{R}_+$ the convolution $K_1 * f$ is defined for $x \in E$, $t \in \mathbb{R}_+$ as

$$K_1 * f(x,t) \doteq \int_E \int_0^t f(y, t-s) K_1(x, dy \times ds).$$

For a substochastic kernel $D$ we define $U^D$ as $U^D = \sum_{k=1}^{\infty} D^k$.

We denote by $\gamma$ the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$.

We present next the definition of the motion of a PDMP. Let $E^0$ be an open subset of $\mathbb{R}^n$ and let $\partial E^0$ be its boundary. A PDMP is determined by its local characteristics $(X, \lambda, Q)$ where the following hold:

- $X$ is a Lipschitz continuous vector field. $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$, which determines a flow $\Phi(x,t)$ such that $\frac{\partial}{\partial t} \Phi(x,t) = X(\Phi(x,t))$ and $\Phi(x,0) = x$ for all $x \in \mathbb{R}^n$.

Define

$$\Gamma^+ \doteq \{x \in \partial E^0 : x = \Phi(y,t) \text{ for some } y \in E^0, t > 0, \text{ and } \Phi(y,s) \in E^0 \forall s \in [0,t]\},$$

and

$$\Gamma^- \doteq \{x \in \partial E^0 : x = \Phi(y,-t) \text{ for some } y \in E^0, t > 0, \text{ and } \Phi(y,-s) \in E^0 \forall s \in [0,t]\}.$$

$\Gamma^+ \subset \partial E^0$ represents the boundary points at which the flow exits from $E^0$. $\Gamma^- \subset \partial E^0$ is characterized by the fact that the flow starting from a point in $\Gamma^-$ will not leave $E^0$ immediately. Therefore it is natural to define the state space for the PDMP by $E \doteq E^0 \cup \Gamma^- - \Gamma^- \cap \Gamma^+$. For all $x \in E$, let us denote $t_+(x) \doteq \inf\{t > 0 : \Phi(x,t) \in \partial E^0\}$, with the convention $\inf\emptyset = \infty$.

- The jump rate $\lambda : E \rightarrow \mathbb{R}_+$ is assumed to be a measurable function satisfying $(\forall x \in E) (\exists \varepsilon > 0) \text{ such that } \int_0^{\varepsilon} \lambda(\Phi(x,s))ds < \infty$.

- $Q : E \cup \Gamma^+ \times \mathcal{B}(E) \rightarrow [0,1]$ is a transition measure satisfying the following property: $(\forall x \in E \cup \Gamma^+) Q(x, E - \{x\}) = 1$.

From these characteristics, it can be shown [6, pp. 62-66] that there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \{P_x\}_{x \in E})$ such that the motion of the process $\{X(t)\}$ starting from a point $x \in E$ may be constructed as follows. Take a random variable $T_1$ such that

$$P_x(T_1 > t) \doteq \begin{cases} e^{-\Lambda(x,t)} & \text{for } t < t_+(x), \\ 0 & \text{for } t \geq t_+(x), \end{cases}$$
where for \( x \in E \) and \( t \in [0, t_*(x)] \)

\[
\lambda(x, t) = \int_0^t \lambda(\Phi(x, s))ds.
\]

If \( T_1 \) generated according to the above probability is equal to infinity, then for \( t \in \mathbb{R}_+ \), \( X(t) = \Phi(x, t) \). Otherwise select independently an \( E \)-valued random variable (labeled \( X_1 \)) having distribution \( Q(\Phi(x, T_1), \cdot) \). The trajectory of \( \{X(t)\} \) starting at \( x \), for \( t \leq T_1 \), is given by

\[
X(t) = \begin{cases} 
\Phi(x, t) & \text{for } t < T_1, \\
X_1 & \text{for } t = T_1.
\end{cases}
\]

Starting from \( X(T_1) = X_1 \), we now select the next interjump time \( T_2 - T_1 \) and post-jump location \( X(T_2) = X_2 \) in a similar way.

This gives a strong Markov process \( \{X(t)\} \) with jump times \( \{T_k\}_{k \in \mathbb{N}} \) (where \( T_0 = 0 \)). The transition semigroup of the process \( \{X(t)\} \) is denoted by \( \{P^t\}_{t \in \mathbb{R}_+} \). We denote by \( \{F_\cdot^{X_k}\}_{k \in \mathbb{R}_+} \) the filtration generated by the process \( \{X(t)\} \).

It is assumed throughout that for all \( (t, x) \in \mathbb{R}_+ \times E \), \( E \mathbb{P}[\sum_{k \leq t} \mathbf{1}_{\{T_k \leq t\}}] < \infty \), implying in particular that \( T_k \to \infty \) as \( k \to \infty \). This is a standard assumption; see, for example, (24.4) or (24.8) in [6].

Now let us introduce the substochastic kernels \( H \) and \( J \) and the Markov kernel \( G \):

\[
H(x, A) \doteq \int_0^{t_*(x)} e^{-(s+\Lambda(x, s))} \mathbf{1}_A(\Phi(x, s))ds,
\]

\[
J(x, A) \doteq \int_0^{t_*(x)} \lambda(\Phi(x, s))e^{-(s+\Lambda(x, s))}Q(\Phi(x, s), A)ds + e^{-\{t_*(x)+\Lambda(x, t_*(x))\}}Q(\Phi(x, t_*(x)), A),
\]

\[
G(x, A) \doteq J(x, A) + H(x, A).
\]

In [8], it was shown that \( G \) as defined in (2.4) is a Markov kernel.

The resolvent kernel associated to the process \( \{X(t)\}_{t \in \mathbb{R}_+} \) is denoted by

\[
R(x, A) \doteq \int_0^\infty P^t(x, A)e^{-t}dt.
\]

As shown in [8], \( R \) can be written in terms of \( H \) and \( J \) as follows:

\[
R = \sum_{j=0}^\infty J^jH.
\]

Let \( \{\Theta_n\} \) (respectively, \( \{Y_n\} \)) be the Markov chain associated to the Markov kernel \( G \) (respectively, \( R \)). In Theorem 3.1 below it will be shown how the Markov chain \( \{\Theta_n\} \) can be generated from the sample paths of the PDMP \( \{X(t)\} \).

In what follows we will present some definitions considering a discrete-time Markov chain \( \{\chi_n\} \) with Markov kernel \( S \) that could be either \( \{\Theta_n\} \) (with \( S = G \)) or \( \{Y_n\} \) (with \( S = R \)). The first return time of a set \( A \in \mathcal{B}(E) \) for the PDMP \( \{X(t)\} \) and for the Markov chain \( \{\chi_n\} \) are defined, respectively, as follows:

\[
\tau^X_A \doteq \inf\{t > 0 : X(t) \in A\}, \quad \tau^S_A \doteq \inf\{n \geq 1 : \chi_n \in A\}.
\]
Associated to these first return times are the return time probability of a set $A \in \mathcal{B}(E)$ for the PDMP $\{X(t)\}$ and for the Markov chain $\{\chi_n\}$, given, respectively, by

$$L^X(x, A) \doteq P_x(\tau^X_A < \infty), \quad L^S(x, A) \doteq P_x(\tau^S_A < \infty).$$

The number of visits to a set $A$ is defined for the PMDP $\{X(t)\}$ and for the Markov chain $\{\chi_n\}$, respectively, as

$$\eta^X_A \overset{\triangle}{=} \int_0^\infty 1_A(X(t))dt, \quad \eta^S_A \overset{\triangle}{=} \sum_{n=1}^{\infty} 1_A(\chi_n).$$

If $F$ is a probability distribution on $\mathbb{R}_+$ (respectively, $b$ is a probability on $\mathbb{N}^*$), then the stochastic kernel $K^X_F$ (respectively, $K^S_b$) associated to $\{X(t)\}$ (respectively, $\{\chi_n\}$) is defined on $E \times \mathcal{B}(E)$ by

$$(\forall x \in E) \quad (\forall A \in \mathcal{B}(E)) \quad K^X_F(x, A) \doteq \int_0^\infty P^t_x(x, A)F(dt),$$

$$(\forall x \in E) \quad (\forall A \in \mathcal{B}(E)) \quad K^S_b(x, A) \doteq \sum_{k=0}^{\infty} b(k)S^k(x, A).$$

A set $C \in \mathcal{B}(E)$ is called a petite set for $\{X(t)\}$ (respectively, $\{\chi_n\}$) if there exist a probability distribution $F$ on $\mathbb{R}_+$ (respectively, a probability $b$ on $\mathbb{N}^*$) and a non-trivial measure $\nu$ on $(E, \mathcal{B}(E))$ such that $(\forall A \in \mathcal{B}(E)) (\forall x \in C) \ K^X_F(x, A) \geq \nu(A)$ (respectively, $(\forall A \in \mathcal{B}(E)) (\forall x \in C) \ K^S_b(x, A) \geq \nu(A)$).

A positive measure $\mu$ (respectively, $\pi$) is called an invariant for the PMDP $\{X(t)\}$ (respectively, for the Markov chain $\{\chi_n\}$) if it is a $\sigma$-finite measure satisfying $\mu = \mu P^t_x$ for all $t \geq 0$ (respectively, $\pi = \pi S$). The PMDP $\{X(t)\}$ is said to be ergodic if it has an invariant probability measure $\mu$ such that

$$(\forall x \in E) \quad \lim_{t \to \infty} \|P^t_x(\cdot) - \mu(\cdot)\| = 0,$$

where $\|\cdot\|$ denotes the total variation norm.

The following definitions apply for both the continuous-time as well as the discrete-time processes, and therefore we suppress the superscript $X$ or $S$. A Markov process is called $\varphi$-irreducible (and $\varphi$ is an irreducibility measure) if for some $\sigma$-finite measure $\varphi$ we have that $E_x(\varphi_A) > 0$ for all $x \in E$ whenever $\varphi(A) > 0$. A set $A \in \mathcal{B}(E)$ is said to be full if $\varphi(A^c) = 0$. An irreducibility measure $\psi$ is called maximal irreducible if for any other $\varphi$ irreducibility measure we have that $\psi \gg \varphi$. A Markov process is called recurrent if for some $\sigma$-finite measure $\varphi$ we have that $E_x(\varphi_A) = \infty$ for all $x \in E$ whenever $\varphi(A) > 0$, and is called Harris recurrent if $E_x(\varphi_A) = \infty$ is replaced by $P_x(\varphi_A = \infty) = 1$. If the Markov process is Harris recurrent, then there exists a unique (up to constant multiples) invariant measure. The Markov process is said to be positive Harris recurrent if it is Harris recurrent and the invariant measure is finite.

3. Preliminary results. In this section we present some preliminary results that will be very important in characterizing the recurrence and Harris recurrence structure of the PDMP $\{X(t)\}$. First, in Theorem 3.1 the Markov chain $\{\Theta_n\}$ generated by the kernel $G$ is shown to be related to the sample path of the PDMP. Since the proof of Theorem 3.1 is rather long and technical, it is presented in an independent section near the end of the paper (see section 7). It is interesting to note that $\{\Theta_n\}$ corresponds to a sampling of the continuous-time process $\{X(t)\}$ at random.
times that depends on a combination of a sequence of independent and identically distributed exponential times with the sequence \( \{T_k\} \) of jump times of the PDMP \( \{X(t)\} \). Moreover it must be pointed out that the Markov kernel \( G \) does not correspond to a generalized resolvent, as studied in the fundamental paper of Meyn and Tweedie [11]. An easy consequence of Theorem 3.1 presented in Corollary 3.2 is that if the first return time of the Markov chain \( \{\Theta_n\} \) to a set \( A \) is finite, then the first return time of the PDMP \( \{X(t)\} \) to the same set \( A \) is finite. Consequently, it will be easy to deduce from this result that if the Markov chain \( \{\Theta_n\} \) is Harris recurrent, then so is the process \( \{X(t)\} \). The last two theorems of this section show that

- the probability of the first return time of \( \{\Theta_n\} \) to a set \( A \) to be finite is bounded below by the probability of the first return time of \( \{\Upsilon_n\} \) to the same set \( A \) to be finite (see Theorem 3.3);
- the average number of visits of \( \{\Theta_n\} \) to a set \( A \) is bounded below by the average number of visits of the Markov chain \( \{\Upsilon_n\} \) (generated by the resolvent) to the same set \( A \) (see Theorem 3.4).

Theorem 3.4 will be used in the next section to show that if the process \( \{X(t)\} \) is recurrent, then so is the Markov chain \( \{\Theta_n\} \). An important consequence of Theorem 3.3 is that if the process \( \{X(t)\} \) is Harris recurrent, then so is the Markov chain \( \{\Theta_n\} \). Theorem 3.3 is surprising and far from trivial to show. The last section of the paper (section 8) is devoted to its proof.

We have the following result, which shows how the Markov chain \( \{\Theta_n\} \) could be generated from the sample path realizations of \( \{X(t)\} \).

**Theorem 3.1.** On the probability space \((\Omega, \mathcal{F}_t, \mathcal{F}, P_x)\) let \( \{s_n\}_{n \geq 0} \) be a sequence of independent and identically distributed \( \mathbb{R}_+ \)-valued random variables with exponential distribution with parameter equal to 1 such that \( \forall t \geq 0 \mathcal{F}_t^X \) and \( \sigma \{s_k : k \geq 0\} \) are independent. Let the sequence of stopping times \( \{\tau_n\}_{n \in \mathbb{N}} \) be defined as follows: \( \tau_0 = 0 \) and

\[
\tau_{n+1} = \sum_{k=0}^{n} 1_{\{T_k \leq \tau_n < T_{k+1}\}} \left[ (\tau_n + s_{n+1}) \wedge T_{k+1} \right].
\]

Then \( \{X(\tau_n)\} \) is a Markov chain with transition probability given by \( G \).

The proof of this result is presented in section 7.

**Remark 3.1.** Roughly speaking the sequence \( \{\tau_n\}_{n \in \mathbb{N}} \) is defined as

\[
\tau_1 = 1_{\{\tau_0 \leq \tau_0 < T_1\}} \left[ (\tau_0 + s_1) \wedge T_1 \right]
= \begin{cases} s_1 \quad \text{if} \ s_1 < T_1, \\ T_1 \quad \text{otherwise}, \end{cases}
\]

\[
\tau_2 = 1_{\{\tau_0 \leq \tau_1 < T_1\}} \left[ (\tau_1 + s_2) \wedge T_2 \right] + 1_{\{\tau_1 \leq \tau_1 < T_2\}} \left[ (\tau_1 + s_2) \wedge T_2 \right]
= \begin{cases} \tau_1 + s_2 \quad \text{if} \ \tau_1 < T_1 \ (\text{equivalently} \ s_1 < T_1), \ and \ \tau_1 + s_2 < T_1, \\ T_1 \quad \text{if} \ \tau_1 < T_1 \ (\text{equivalently} \ s_1 < T_1), \ and \ \tau_1 + s_2 \geq T_1, \\ T_1 + s_2 \quad \text{if} \ \tau_1 = T_1 \ (\text{equivalently} \ s_1 \geq T_1), \ and \ T_1 + s_2 < T_2, \\ T_2 \quad \text{otherwise}, \end{cases}
\]

and so on.
Without loss of generality, it can be considered that $\Theta_n = X(\tau_n)$, since $\{\Theta_n\}$ was defined in section 2 as a Markov chain generated by $G$, and, from the previous theorem, that $\{\Theta_n\}$ and $\{X(\tau_n)\}$ have the same probability distribution. An important corollary of the previous theorem is the following inclusion for the first return times of the Markov chain $\{\Theta_n\}$ and the process $\{X(t)\}$.

**Corollary 3.2.** For any set $A \in \mathcal{B}(E)$,

$$\{\tau^G_A < \infty\} \subset \{\tau^X_A < \infty\}.$$  \hspace{1cm} (3.2)

**Proof.** This is a straightforward consequence from the fact that we can consider $\Theta_n = X(\tau_n)$, as shown in Theorem 3.1. \qed

We have the following important theorem establishing a link between the probability of the first return time to a set for the Markov chains $\{\Theta_n\}$ and $\{\Upsilon_n\}$.

**Theorem 3.3.** For every $x \in E$ and for $A \in \mathcal{B}(E)$,

$$L^G(x, A) \geq L^R(x, A).$$  \hspace{1cm} (3.3)

The proof of this result is presented in section 8.

Combining (3.2) and (3.3) we have, for every $x \in E$ and $A \in \mathcal{B}(E)$, the following important inequalities:

$$L^R(x, A) \leq L^G(x, A) \leq L^X(x, A).$$

We conclude this section with the following theorem, providing a link between the average numbers of visits for the Markov chains generated by the kernel $G$ and $R$.

**Theorem 3.4.** For every $x \in E$ and for $A \in \mathcal{B}(E)$,

$$U^G(x, A) \geq U^R(x, A).$$  \hspace{1cm} (3.4)

**Proof.** From Lemma 4.1 in [17], we have $G^n = (J+H)^n = J^n + \sum_{j=1}^{n} J^{j-1} H G^{n-j}$ for any $n \in \mathbb{N}^*$. Consequently, it follows that

$$U^G = U^J + \sum_{n=1}^{\infty} \sum_{j=1}^{n} J^{j-1} H G^{n-j} 1_{\{j \leq n\}}$$

$$= U^J + R + RU^G,$$

where the last equality follows from (2.6). Hence, by iteration on $n$ in the previous equation, we get that for all $n \geq 1$,

$$U^G = U^J + \sum_{j=1}^{n-1} R^j U^J + \sum_{j=1}^{n} R^j + R^n U^G \geq \sum_{j=1}^{n} R^j.$$

Taking the limit as $n$ goes to $\infty$, we get the result. \qed

4. Characterization of the recurrence and Harris recurrence structures of the PDMP in terms of the Markov kernel $G$. The aim of this section is to characterize the (Harris) recurrence properties between the PMDP $\{X(t)\}$ and the Markov chain $\{\Theta_n\}$ generated by the kernel $G$. First, it is proved in Proposition 4.1 that $\{X(t)\}$ is irreducible if and only if $\{\Theta_n\}$ is irreducible. Then a generalization of Theorem 3.5 in [8] is presented in Theorem 4.2 giving a one-to-one correspondence between the invariant (positive and $\sigma$-finite) measures for the PDMP $\{X(t)\}$ and the
Markov chain \( \{\Theta_n\} \). Using the preliminary results derived in the previous section, it is shown in Theorems 4.4 and 4.6 that the PDMP \( \{X(t)\} \) is recurrent (respectively, Harris recurrent) if and only if the Markov chain \( \{\Theta_n\} \) is recurrent (respectively, Harris recurrent). One would expect a natural generalization of such equivalence results for positivity between the processes \( \{X(t)\} \) and \( \{\Theta_n\} \). In fact, this result does not hold. Indeed, it is shown in Corollary 4.5 that the positive recurrence of the process \( \{X(t)\} \) is equivalent to a weaker form of stability for the Markov chain \( \{\Theta_n\} \). Namely, \( \{X(t)\} \) is positive recurrent if and only if \( \{\Theta_n\} \) is recurrent and its unique invariant measure \( \pi \) satisfies the condition given by \( \pi H(E) < \infty \), which is far less demanding than positive recurrence for \( \{\Theta_n\} \). A similar result will be proved for the positive Harris recurrence of \( \{X(t)\} \) (see Corollary 4.7).

We have the following proposition characterizing the irreducibility of the process \( \{X(t)\} \) and the Markov chain \( \{\Theta_n\} \).

**Proposition 4.1.** The PDMP \( \{X(t)\} \) is irreducible if and only if the Markov chain \( \{\Theta_n\} \) is irreducible.

**Proof.** Suppose that the Markov chain \( \{\Theta_n\} \) is \( \varphi \)-irreducible. From Proposition 4.2.1 in [12, p. 87], whenever \( \varphi(A) > 0 \) for any \( A \in \mathcal{B}(E) \) we have that \( L^G(x, A) > 0 \) for all \( x \in E \). From (3.2) we have that \( L^N(x, A) > 0 \) for all \( x \in E \) whenever \( \varphi(A) > 0 \) for \( A \in \mathcal{B}(E) \). By using Proposition 2.1 in [13], it follows that the PDMP \( \{X(t)\} \) is \( \mu \)-irreducible with \( \mu = \varphi R \), where we recall that \( R \) is the resolvent defined in (2.5).

Now suppose that \( \{X(t)\} \) is \( \Psi \)-irreducible. Then for \( A \in \mathcal{B}(E) \) with \( \Psi(A) > 0 \), we have, for all \( x \in E \), \( E_x \pi_0^\Psi > 0 \). Since \( E_x \pi_0^\Psi = U^{\Psi}(x, A) \) (see, for instance, [19]), it implies that \( L^R(x, A) > 0 \). From Theorem 3.3, we get the result. \( \square \)

Recall that by definition an invariant measure is always \( \sigma \)-finite and positive. The next result shows that there exists a one-to-one correspondence between the set of invariant measures for the PDMP \( \{X(t)\} \) and the set of invariant measures for the Markov chain \( \{\Theta_n\} \) generated by \( G \). It extends Theorem 3.5 in [8], which was restricted to the set of invariant probability measures for the PDMPs.

**Theorem 4.2.** (i) If \( \mu \) is an invariant measure for \( \{X(t)\} \), then \( \mu \sum_{j=0}^\infty J^j H = \mu \).

(ii) If \( \pi \) is an invariant measure for \( \{\Theta_n\} \), then \( \pi H \) is invariant for \( \{X(t)\} \) and \( \pi H \sum_{j=0}^\infty J^j = \pi \).

**Proof.** Let us show (i). Let \( \mu \) be an invariant measure for \( \{X(t)\} \) and set \( \pi = \sum_{j=0}^\infty \mu J^j \). Let us show that \( \pi \) is \( \sigma \)-finite. Since \( \mu \) is \( \sigma \)-finite, there exists a partition \( \{A_i\} \) of \( E \) such that \( \mu(A_i) < \infty \). Define \( C_n = \bigcup_{i=1}^n A_i \) and \( B_{n,m} = \{y \in E : H(y, C_n) > \frac{1}{m} \} \) for \( m \in \mathbb{N}^* \). Notice now that for every \( x \in E \), we have that \( 0 < H(x, E) < 1 \), and so \( \bigcup_{n,m} B_{n,m} = E \). From (2.6) we have that

\[
\mu > \mu(C_n) = \int_E H(y, C_n) \pi_0(dy) = \int_{B_{n,m}} H(y, C_n) \pi_0(dy) \geq \frac{1}{m} \pi(B_{n,m}),
\]

showing that \( \pi \) is \( \sigma \)-finite. Finally notice from (2.4) and (2.6) that

\[
\pi G = \pi J + \pi H = \sum_{j=1}^\infty \mu J^j + \mu H = \sum_{j=1}^\infty \mu J^j + \mu = \sum_{j=0}^\infty \mu J^j = \pi,
\]

showing that \( \pi \) is invariant for \( \{\Theta_n\} \). Moreover,

\[
\mu \sum_{j=0}^\infty J^j H = \mu R = \mu,
\]
completing the proof of (i).

Let us now show (ii). Let \( \pi \) be an invariant measure for \( \{ \Theta_n \} \). For any \( n \in \mathbb{N}^* \), we have

\[
\pi \sum_{j=1}^{n} J^j H + \pi H = \pi G \sum_{j=0}^{n} J^j H = \pi \sum_{j=1}^{n} J^j H + \pi J^{n+1} H + \pi H \sum_{j=0}^{n} J^j H.
\]

In order to cancel out the identical term \( \pi \sum_{j=1}^{n} J^j H \) on both sides of the previous equation, one must first check that all the measures under consideration are \( \sigma \)-finite.

Since \( \pi = \pi G = \pi H + \pi J \), it can be shown easily by induction that \( \pi J^H \) is \( \sigma \)-finite for all \( j \in \mathbb{N} \). Consequently, for all \( j \in \mathbb{N}^* \), the measures \( \pi \sum_{j=1}^{n} J^j H \), \( \pi H \), \( \pi J^{n+1} H \), and \( \pi H \sum_{j=0}^{n} J^j H \) are \( \sigma \)-finite, implying that

\[
\pi H = \pi J^{n+1} H + \pi H \sum_{j=0}^{n} J^j H.
\]

Moreover, it can be shown that \( J^n(x, A) = E_x \left[ e^{-T_n} 1_A(X(T_n)) \right] \) for all \( n \in \mathbb{N}^* \), \( x \in E \), and \( A \in \mathcal{B}(E) \). By using the dominated convergence theorem and the fact that \( \lim_{n \to \infty} T_n = +\infty \), it follows that for all \( A \in \mathcal{B}(E) \)

\[
\lim_{n \to \infty} \pi J^n(A) = 0.
\]

Combining (4.1) and (4.2), we have that \( \mu = \pi H = \pi H \sum_{j=0}^{\infty} J^j H = \mu R \), and from Lemma 1 in [1] it follows that \( \mu \) is an invariant measure for \( \{ X(t) \} \). Now we have that

\[
\pi H \sum_{j=0}^{n} J^j + \pi J^{n+1} + \pi \sum_{j=1}^{n} J^j = \pi + \pi \sum_{j=1}^{n} J^j.
\]

It follows that \( \pi H \sum_{j=0}^{n} J^j + \pi J^{n+1} = \pi \) by using the same arguments as above. Equation (4.2) yields that \( \lim_{n \to \infty} \pi H \sum_{j=0}^{n} J^j = \pi \), showing (ii).

**Remark 4.1.** A straightforward consequence of Theorem 4.2 is the following result: There exists a finite invariant measure for \( \{ X(t) \} \) if and only if there exists an invariant measure \( \pi \) for \( \{ \Theta_n \} \) satisfying \( \pi H(E) < \infty \). Note that this result was already proved in Theorem 3.5 in [8].

The next two results show that if the PDMP is recurrent, then so is the Markov chain generated by \( G \), and vice versa.

**Proposition 4.3.** If \( H \in \mathcal{B}(E) \) is recurrent for the process \( \{ X(t) \} \), then \( H \) is recurrent for the Markov chain \( \{ \Theta_n \} \).

**Proof.** If \( H \in \mathcal{B}(E) \) is recurrent for \( \{ X(t) \} \), then there exists a measure \( \nu \) on \( (H, \mathcal{B}(H)) \) such that for all \( A \in \mathcal{B}(H) \) with \( \nu(A) > 0 \), \( E_x \left[ \eta^X_0 \right] = U^R(x, A) = \infty \) for every \( x \in H \). From Theorem 3.4, it follows that \( U^G(x, A) = \infty \), showing the result.

**Theorem 4.4.** The PDMP \( \{ X(t) \} \) is recurrent if and only if the Markov chain \( \{ \Theta_n \} \) associated to \( G \) is recurrent.

**Proof.** Suppose that \( \{ X(t) \} \) is transient and \( \{ \Theta_n \} \) is recurrent. Let \( \varphi \) be a maximal irreducibility measure for \( \{ \Theta_n \} \). Then from Proposition 9.0.1 in [12], \( E = H \cup T \), where \( T \in \mathcal{B}(E) \) is \( \varphi \)-null and transient for \( \{ \Theta_n \} \). \( H \in \mathcal{B}(E) \) is nonempty and absorbing for \( \{ \Theta_n \} \), and every subset of \( H \) in \( \mathcal{B}(E)^+ = \{ A \in \mathcal{B}(E) : \varphi(A) > 0 \} \).
is Harris recurrent. On the other hand, since \( \{X(t)\} \) is transient, it follows that 
\[
E = \bigcup_{i=1}^{\infty} E_i \text{ with } E_i [\eta_{E_i}^X] \leq M_i < \infty \text{ for every } x \in E.
\]
Define \( A_n \doteq \bigcup_{i=1}^{n} E_i \cap H \).
Consider \( k \in \mathbb{N}^+ \) such that \( \varphi(A_k) > 0 \).
For every \( x \in E \), we have
\[
E_x [\eta_{A_k}^X] \leq \sum_{i=1}^{k} E_x [\eta_{E_i}^X] \leq M,
\]
with \( M \doteq \sum_{i=1}^{k} M_i \). However, since \( A_k \) is a subset of \( H \) and belongs to \( \mathcal{B}(E)^+ \), the set \( A_k \) is Harris recurrent for \( \{\Theta_n\} \).
Consequently, by using (3.2), we have that for every \( x \in A_k \), \( 1 = L^G(x, A) \leq L^X(x, A) \), contradicting the fact that \( E_x [\eta_{A_k}^X] \leq M \).
Therefore, if \( \{\Theta_n\} \) is recurrent, then \( \{X(t)\} \) is recurrent.

The converse follows from Proposition 4.3, giving the result. \( \square \)

The next corollary emphasizes a split with the previous equivalence results. Indeed it is shown that the process is positive recurrent if and only if \( \{\Theta_n\} \) satisfies a weaker condition: recurrence and a technical condition for its unique (\( \sigma \)-finite) invariant measure.

**Corollary 4.5.** The PDMP \( \{X(t)\} \) is positive recurrent if and only if the Markov chain \( \{\Theta_n\} \) is recurrent with invariant measure \( \pi \) satisfying \( \pi H(E) < \infty \).

**Proof.** The result easily follows from Remark 4.1 and Theorem 4.4. \( \square \)

We prove now that the Harris recurrent properties are equivalent for \( \{X(t)\} \) and \( \{\Theta_n\} \).

**Theorem 4.6.** The PDMP \( \{X(t)\} \) is Harris recurrent if and only if the Markov chain \( \{\Theta_n\} \) is Harris recurrent.

**Proof.** Suppose that the Markov chain \( \{\Theta_n\} \) is Harris recurrent. Denote by \( \Psi^G \) a maximal irreducibility measure for the Markov chain \( \{\Theta_n\} \). Then for any set \( A \in \mathcal{B}(E) \) satisfying \( \Psi^G(A) > 0 \) it follows from Corollary 3.2 that \( 1 = L^G(x, A) \leq L^X(x, A) \) for all \( x \in E \). Therefore, \( \{X(t)\} \) is Harris recurrent by using Theorem 1.1 in [11].

Now assume that the PDMP \( \{X(t)\} \) is Harris recurrent. From the equivalence results in [19], if the PDMP \( \{X(t)\} \) is Harris recurrent, then the Markov chain \( \{\Upsilon_n\} \) associated to the resolvent \( R \) is Harris recurrent.
Moreover, by using Proposition 4.1 \( \{\Theta_n\} \) is irreducible. Let us denote by \( \Psi^G \) (respectively, \( \Psi^R \)) a maximal irreducible measure for \( \{\Theta_n\} \) (respectively, \( \{\Upsilon_n\} \)).
According to the definition of Harris recurrence (see [12, p. 200]), we want to show that if \( \Psi^G(A) > 0 \), then \( P_x \left[ \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} \{\Theta_n \in A\} \right] = 1 \) for all \( x \in E \). From (ii) and (iii) of Proposition 5.5.5 in [12], it follows that there exists an increasing sequence of petite sets \( \{C_k\}_{k \in \mathbb{N}} \) for \( \{\Theta_n\} \) such that \( E = \bigcup_{k \in \mathbb{N}} C_k \) with \( \Psi^G(C_k) > 0 \) and \( \Psi^R(C_k) > 0 \) for all \( k \in \mathbb{N} \). Since \( \{\Upsilon_n\} \) is Harris recurrent we have for all \( x \in \mathbb{N} \) that \( L^R(x, C_k) = 1 \) for all \( x \in C_k \).
From Theorem 3.3 we have that \( L^G(x, C_k) = 1 \) for all \( x \in C_k \), and from Proposition 9.1.1 in [12], it follows that \( C_k \) is Harris recurrent for \( \{\Theta_n\} \). The remainder of the proof follows the same steps as the end of the proof of Theorem 9.1.4 in [12], and it will be presented for the sake of completeness. From Lemma 5.5.1 in [12], we have that for all \( A \in \mathcal{B}(E) \) with \( \Psi^G(A) > 0 \) there exists \( \delta > 0 \) such that \( \inf_{x \in C_k} L^G(x, A) > \delta \).
However, \( C_k \) is Harris recurrent for \( \{\Theta_n\} \), and from Theorem 9.1.3(i) in [12], we have that for all \( x \in C_k \), \( P_x \left[ \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} \{\Theta_n \in A\} \right] = 1 \). The result follows after recalling that \( E = \bigcup_{k \in \mathbb{N}} C_k \). \( \square \)

**Remark 4.2.** In the previous proof note that if \( A \) is a set such that \( \psi^G(A) > 0 \), then it does not necessarily imply that \( \psi^R(A) > 0 \). That is why we needed to proceed.
through the tool of petite sets.

As for the positive recurrence property, the following result points out the split with the previous theorem by showing that the positive Harris recurrence is equivalent to a weaker form of stability for the chain \( \{ \Theta_n \} \).

**Corollary 4.7.** The PDMP \( \{ X(t) \} \) is positive Harris recurrent if and only if the Markov chain \( \{ \Theta_n \} \) associated to \( G \) is Harris recurrent with invariant measure \( \pi \) satisfying \( \pi H(E) < \infty \).

**Proof.** Combining Remark 4.1 and Theorem 4.6, we obtain the result. \( \square \)

The final result of this section gives a condition for the PDMP to be ergodic. As shown in [13, Theorem 6.1], if \( \{ X(t) \} \) is positive Harris recurrent, then it is ergodic if and only if some skeleton chain is irreducible. Thus from Corollary 4.7 it remains only to obtain a tractable and equivalent condition to show that some skeleton chain is irreducible. Let \( N \) be the kernel defined on \( E \times \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+) \) by

\[
N(x, A \times \Gamma) = \int_0^{t_s(x)} 1_{[s]}(t_s(x)) e^{-\Lambda(x,s)} Q(\Phi(x,s), A) ds
\]

(4.3)

where \( A \in \mathcal{B}(E), \Gamma \in \mathcal{B}(\mathbb{R}_+) \). Define \( M : E \times \mathcal{B}(E) \times \mathbb{R}_+ \to \mathbb{R} \) by

\[
M^t(x, A) = 1_{\{t < t_s(x)\}} 1_A(\Phi(x,t)) e^{-\Lambda(x,t)}
\]

for \( A \in \mathcal{B}(E), t \in \mathbb{R}_+, \) and \( x \in E \). For \( x \in E \) and \( k \in \mathbb{N} \), define the measure \( \alpha^k(x,.\) on \( (E \times \mathbb{R}_+, \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+)) \) by

\[
\alpha^k(x,B) = \int_B N^{*k} * M^t(x,dy) \gamma(dt)
\]

for \( B \in \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+) \).

**Theorem 4.8.** Suppose that the Markov chain \( \{ \Theta_n \} \) is positive Harris recurrent and denote by \( \Psi^G \) a maximal irreducibility measure for \( \{ \Theta_n \} \). The following propositions are equivalent:

(i) There exist a set \( C \in \mathcal{B}(E) \) with \( \Psi^G H(C) > 0 \) such that for all \( x \in C \) there exists \( k \in \mathbb{N} \) for which the measure \( \alpha^k(x,.\) is nonsingular with respect to the measure \( \Psi^G H \otimes \gamma \).

(ii) The PDMP \( \{ X(t) \} \) is ergodic.

**Proof.** From Lemma (27.3) in [6], it is easy to obtain that for all \( x \in E \) and for \( A \in \mathcal{B}(E) \),

\[
P^t(x,A) = \sum_{k=0}^{\infty} N^{*k} * M^t(x,A).
\]

(4.5)

Consequently, for any \( B \in \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+) \), we have

\[
\int_B P^t(x,dy) \gamma(dt) = \sum_{k=0}^{\infty} \alpha^k(x,B).
\]

(4.6)

From Corollary 4.7, the PDMP \( \{ X(t) \} \) is positive Harris recurrent. Denote by \( \mu \) the invariant probability measure for \( \{ X(t) \} \) and by \( \Psi^X \) a maximal irreducibility measure for \( \{ X(t) \} \). From Proposition 4.1 there exists a positive \( \sigma \)-finite measure
(labeled \( \pi \)) invariant for \( G \) satisfying \( \pi H = \mu \). By hypothesis, we also have that \( G \) is irreducible. From Theorem 10.4.9 in [12] we obtain that \( \mu \sim \Psi^X \) and \( \pi \sim \Psi^G \), and thus \( \Psi^G H \sim \Psi^X \).

According to Proposition 6.2 of Niemi and Nummelin [16] and by using (4.6), a skeleton chain is irreducible if and only if there exists a set \( C \) satisfying \( \Psi^X(C) > 0 \) and such that for all \( x \in C \) there exists an integer \( k \) for which the measure \( \alpha^k(x, \cdot) \) is nonsingular with respect to the measure \( \Psi^X \otimes \gamma \). However, from Theorem 6.1 in [13] and by using the fact that \( \Psi^G H \sim \Psi^X \), we have that the PDMP \( \{X(t)\} \) is ergodic if and only if item (i) is satisfied, showing the result. 

5. **Sufficient conditions for stability and ergodicity of PDMPs.** Based on the results derived in the previous section we now present some sufficient conditions for the existence of an invariant probability measure, positive Harris recurrence, and ergodicity for the PDMP \( \{X(t)\} \). These various stability conditions are inspired by the results obtained in [12, 15] and are based on modified Foster–Lyapunov criteria through the Markov kernel \( G \). As mentioned before, this will lead to tractable criteria since the kernel \( G \) can be explicitly characterized in terms of the three local characteristics of the PDMP. It must be pointed out that the results derived in [12, 15] are not directly applicable to our case mainly because we need to obtain a criterion ensuring the existence of \( \sigma \)-finite invariant measure for \( G \) satisfying \( \pi H(E) < \infty \).

We present next a Foster–Lyapunov criterion, based on the stochastic kernel \( G \), for obtaining an invariant probability measure \( \mu \) for the PDMP \( \{X(t)\} \) satisfying \( \int_E f(x)\mu(dx) < \infty \), for any measurable function \( f \geq 1 \). This result generalizes Corollary 4.5 in [8] by relaxing the hypotheses on the Lyapunov function \( V \) and the test set \( D \), and allowing us to consider the measurable function \( f \geq 1 \).

**Proposition 5.1.** Let \( f : E \to [1, \infty) \) be a measurable function. Suppose that the Markov kernel \( G \) is recurrent and that the following Foster–Lyapunov criterion is satisfied:

\[
(5.1) \quad (\forall x \in E) \quad GV(x) \leq V(x) - Hf(x) + bI_D(x),
\]

where \( D \) is a petite set for the Markov chain associated to \( G \), and \( V : E \to [0, \infty] \) is a measurable function (with \( V(x_0) < +\infty \) for at least one \( x_0 \in E \)). Then there exists a unique invariant probability measure \( \mu \) for the PDMP \( \{X(t)\} \) and moreover \( \int_E f(x)\mu(dx) < \infty \).

**Proof.** Let \( \Psi^G \) be a maximal irreducibility measure for \( G \). From Proposition 5.5.5 in [12], there exists a sequence of petite sets, say \( \{C_n\} \), such that \( C_n \uparrow E \). Define \( B_n = C_n \cap \{x \in E : V(x) \leq n\} \). Notice that the set \( A = \{x \in E : V(x) < +\infty\} \) is absorbing (indeed, from (5.1), if \( x \in A \), then \( GV(x) < \infty \) and thus \( G(x, A) = 1 \)), and thus from Proposition 4.2.3 in [12], it is full. It follows that there exists \( k \in \mathbb{N} \) such that \( \Psi^G(B_k) > 0 \). Clearly, \( B_k \) is a petite set for \( G \), and \( V \) is bounded in \( B_k \). Now applying Theorem 4.2.3 in [12] we obtain that \( \sup_{x \in B_k} E_x \left[ \sum_{j=0}^{k-1} Hf(\Theta_j) \right] < +\infty \).

By using the same arguments as in Theorem 4.4 of [8], it follows that there exists a unique invariant measure \( \pi \) of \( G \) which, moreover, satisfies \( \int_E f(x)\pi H(dx) < \infty \). Consequently, it follows from Theorem 4.2 and Remark 4.1 that the measure \( \mu \) defined by \( \pi H \) is finite, invariant for the PDMP, and satisfies \( \int_E f(x)\mu(dx) < \infty \). We get the uniqueness by using Proposition 4.1 since \( G \) is irreducible, showing the result.

Next we present a theorem that requires only \( \varphi \)-irreducibility of the Markov kernel \( G \), replacing the recurrence condition by another Lyapunov criterion. Moreover this result shows that the PDMP \( \{X(t)\} \) is positive Harris recurrent.
Theorem 5.2. Let \( f : E \to [1, \infty) \) be a measurable function. Suppose that the Markov kernel \( G \) is irreducible and satisfies the following Lyapunov criterion:

\[
\begin{align*}
(5.2) & \quad (\forall x \in C^c) \quad GW(x) \leq W(x), \\
(5.3) & \quad (\forall x \in E) \quad GV(x) \leq V(x) - Hf(x) + bI_D(x),
\end{align*}
\]

where \( C \) and \( D \) are petite sets for the Markov chain associated to \( G \), \( V : E \to [0, \infty] \) is a measurable function (with \( V(x_0) < +\infty \) for at least one \( x_0 \in E \)), and \( W \) is a function unbounded off petite sets. Then the PDMP \( \{X(t)\} \) is positive Harris recurrent and its unique invariant probability measure \( \mu \) satisfies \( \int_E f(x)\mu(dx) < \infty \).

Proof. From Theorem 9.1.8 in [12], inequality (5.2) implies that \( G \) is Harris recurrent, showing that the PDMP is Harris recurrent from Theorem 4.6. From Proposition 5.1 we get the result. \( \square \)

Next we present a state-dependent criterion based on Theorem 2.2 of [15]. For this consider a random variable \( \varsigma \) on \( \mathbb{N}^* \) for each \( x \in E \), independent of \( \{\Theta_k\} \), and with distribution \( a_x(k) \), \( k \in \mathbb{N}^* \). Recall that \( K_{a_x}^G(x,A) = \sum_{k=1}^{\infty} a_x(k)G^K(x,A) \) for any \( A \in \mathcal{B}(E) \). Suppose that \( \sum_{i>k} a_x(i) \leq B_2a_x(k) \). We have the following theorem.

Theorem 5.3. Let \( f : E \to [1, \infty) \) be a measurable function. Suppose that the Markov kernel \( G \) is \( \varphi \)-irreducible and satisfies the following Lyapunov criterion:

\[
\begin{align*}
(5.4) & \quad (\forall x \in C^c) \quad K_{a_x}^G W(x) \leq W(x), \\
(5.5) & \quad (\forall x \in E) \quad K_{a_x}^G V(x) \leq V(x) - Hf(x) + bI_D(x),
\end{align*}
\]

where \( C \) and \( D \) are petite sets for the Markov chain associated to \( G \), \( V : E \to [0, \infty] \) is a measurable function bounded on \( D \), and \( W \) is a nonnegative function unbounded off petite sets. Then the PDMP \( \{X(t)\} \) is positive Harris recurrent and its unique invariant probability measure \( \mu \) satisfies \( \int_E f(x)\mu(dx) < \infty \).

Proof. From Theorem 2.2(i) in [15], inequality (5.4) implies that \( G \) is Harris recurrent. From (5.5) and the proof of Theorem 2.2(i) in [15, p. 158], we have that for some nonnegative function \( W(x) \), with \( W(x) < \infty \) for all \( x \in E \), that \( GW(x) \leq W(x) - Hf(x) + bI_D(x) \) for all \( x \in E \). From Theorem 5.2 we get the desired result. \( \square \)

Finally we present a sufficient tractable condition for the PDPM to be ergodic.

Theorem 5.4. Suppose that the hypotheses of Theorem 5.2 and item (i) of Theorem 4.8 are satisfied. Then the PDMP \( \{X(t)\} \) is ergodic.

Proof. The result follows by combining Theorems 4.8 and 5.2. \( \square \)

Remark 5.1. The results have been developed considering the stochastic kernel \( G \) defined on the measurable state space \( (E,\mathcal{B}(E)) \). It may be convenient in some applications to consider the extended state space \( E \cup \Gamma^+ \) instead of \( E \), with the stochastic kernel \( G \) extended to include the points in \( \Gamma^+ \). Let us define the kernel \( G \) on \( (E \cup \Gamma^+,\mathcal{B}(E \cup \Gamma^+)) \) (corresponding to an extension of \( G \)) by \( (\forall x \in E \cup \Gamma^+) (\forall A \in \mathcal{B}(E \cup \Gamma^+)) \)

\[
\overline{G}(x,A) = 1_E(x)G(x,A \cap E) + 1_{\Gamma^+}(x)Q(x,A \cap E).
\]

It is easy to see that \( \overline{G} \) is a stochastic kernel on \( (E \cup \Gamma^+,\mathcal{B}(E \cup \Gamma^+)) \). Moreover, it can be easily shown that \( \overline{G} \) has the following important properties:

(a) \( \varphi \) is an irreducibility measure for \( G \) if and only if \( \varphi \) is an irreducibility measure for \( \overline{G} \).

(b) \( \pi \) is an invariant measure for \( G \) if and only if \( \pi \) is an invariant measure for \( \overline{G} \).
(c) $G$ is Harris recurrent if and only if $\mathcal{G}$ is Harris recurrent.

Consequently, the result presented in Theorem 5.4 remains valid if the Markov kernel $G$ is replaced by the Markov kernel $\mathcal{G}$.

6. Example. This example is based on the capacity expansion model, analyzed in [7], [6, Example (34.45)], and [3, section 7]. The existence of an invariant probability measure for a generalized version of this model was studied in [8]. In the present work this example is revisited and the result of [8] is strengthened: Proposition 6.1 gives sufficient conditions to ensure that this general capacity expansion model is ergodic.

Capacity expansions are general processes of adding facilities to meet a rising demand. Each project meets demand. The demand for some utility is modeled as a random point process; i.e., it is a rate

$$
\lambda_i(u) \geq \liminf_{t \to \infty} \lambda_i(u + t) > \limsup_{t \to \infty} \lambda_i(u + t)
$$

exceeds demand. We assume that if there is an excess demand of at least $p$ units, then the construction of a new project is started at a rate $r_i$ per unit of time and is completed after a lead time of $L_i$ units of time. If the excess demand is less than $p$, then no construction takes place. New demand occurs with rate $\lambda_i(u)$, where $u$ is the time spent by the current project. This problem can be modeled as a PDMP $\{x_i\}$ with state space

$$
\mathbb{E} \doteq \{\{p - K_p, \ldots, p - 1\} \times \{0\}\} \cup \bigcup_{n=p}^{\infty} \{n, [0, \mathcal{L}_n]\},
$$

where $p \in \mathbb{N}$ and $(K_i)_{i \geq p}$ is a sequence of integers, and $(\mathcal{L}_i)_{i \geq p}$ is an increasing sequence of strictly positive real numbers. $\mathbb{N}_p$ denotes the set of integers greater than or equal to $p$. The local characteristics are given by

$$
\lambda((i, u)) = \lambda_i(u),
$$

$$
\Phi((i, u), t) = \begin{cases} (i, 0) & \text{for } i \in \{p - K_p, \ldots, p - 1\}, \\
(i, u + r_i t) & \text{for } i \geq p,
\end{cases}
$$

$$
Q((i, u), \{(i + 1, u)\}) = 1 \quad \text{and} \quad Q((i, \mathcal{L}_i), \{(i - K_i, 0)\}) = 1.
$$

Consequently, we have $\Gamma^+ = \{(n, \mathcal{L}_n) : n \geq p\}$. Let us introduce the following assumptions:

(H1) There exists $r > 0$ such that for all $i, r_i \geq r$.

(H2) For $i \geq p$, $i - K_i \geq p - K_p > 0$ and $\lim_{i \to \infty} i - K_i = \infty$.

(H3) There exists an integer $k_0$ such that for all $i \geq k_0$, $\frac{K_i}{\mathcal{L}_i} \leq \frac{K_{i+1}}{\mathcal{L}_{i+1}}$.

(H4) For $i \geq p$, $\lambda_i(u)$ is a continuous real-valued function on $[0, \mathcal{L}_i]$.

(H5) For $i \in \{p - K_p, \ldots, p - 1\}$, $\lambda_i > 0$.

(H6) $\limsup_{i \to \infty} \frac{\mathcal{L}_i}{\mathcal{L}_i} \max_{u \in [0, \mathcal{L}_i]} \lambda_i(u) < 1$.

We have the following result.

**Proposition 6.1.** If (H1)–(H6) hold, then the PDMP $\{X(t)\}$ is ergodic.

**Proof.** Following the proof of Proposition 5.1 in [8], we have that the Markov kernel $\mathcal{G}$ defined on $(\mathbb{E} \cup \Gamma^+, \mathcal{B}(\mathbb{E} \cup \Gamma^+))$ is weak Feller and for the real-valued function defined on $\mathbb{E} \cup \Gamma^+$ by

$$
V((i, u)) = i - \frac{K_i}{\mathcal{L}_i} u,
$$

there exist $(\varepsilon, b) \in \mathbb{R}_+^2$ and an integer $k_0$ such that

$$
\mathcal{G}V((i, u)) - V((i, u)) \leq -\varepsilon H((i, u), \mathbb{E}) + b1_{C_{k_0}}((i, u)),
$$

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where $C_{k_0} \doteq \{p - K_p, \ldots, p - 1\} \times \{0\} \cup \bigcup_{n=0}^{k_0} \{n, [0, \mathcal{L}_n]\}$.

Using the hypotheses, it can be shown that $\delta_{\{(p-1,0)\}}$ is an irreducibility measure for $\{X(t)\}$. The support of $\delta_{\{(p-1,0)\}}$ clearly has a nonempty interior. It follows from item (ii) of Theorem 3.4 in [13] that all compact subsets of $E \cup \Gamma^+$ are petite sets. Therefore, $C_{k_0}$ is a petite set and the function $V$ is unbounded off petite sets. Consequently, the hypotheses of Theorem 5.2 are satisfied.

Now let us show that item (i) of Theorem 4.8 is satisfied for the Markov kernel $\overline{G}$. Define $C \doteq \{(p-1,0)\}$. Clearly, $\delta_{\{(p-1,0)\}}(C) > 0$ and

$$
(\forall x \in C) \forall A \in \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+) \quad \alpha^0(x, A) = \int_A \delta_{\{(p-1,0)\}}(dy)e^{-\lambda_{p-1}(0)s}\gamma(ds),
$$

since $t_*((p-1,0)) = +\infty$ and $\Phi((p-1,0), s) = (p-1, 0)$. Using assumption (H5), we obtain that

$$
(\forall x \in C) \quad \alpha^0(x, \cdot) \sim \delta_{\{(p-1,0)\}} \otimes \gamma.
$$

Consequently, for all $x \in C$, $\alpha^0(x, \cdot)$ is nonsingular with respect to the measure $\delta_{\{(p-1,0)\}} \otimes \gamma$. However, if $\Psi^G$ denotes an irreducibility measure for $\overline{G}$, then $\Psi^G \ll \Psi^H$. Consequently, item (i) of Theorem 4.8 is satisfied for the Markov kernel $\overline{G}$. Combining Remark 5.1 and Theorem 5.4, we obtain the result.

7. Proof of Theorem 3.1. In this section we present the proof of Theorem 3.1. First, Lemma 7.1 gives another expression for the Markov kernel $G$. The following results (see Lemmas 7.2 and 7.4 and Proposition 7.3) will give some relations between the sequence of stopping times $\{\tau_n\}$ and the jump times of the PDMP $\{T_n\}$ that will be used in what follows. The filtration generated by the PDMP stopped at times $\{\tau_n\}$ and at times $\{T_n\}$ are studied in Lemma 7.5. Finally, the proof of Theorem 3.1 follows by combining Proposition 7.6 and the previous results.

**Lemma 7.1.** Assume that $\forall t \geq 0, F^x_t$ and $\sigma\{s_k : k \geq 0\}$ are independent. Then

$$
G(x, A) = E_x\left[1_A[X(T_1 \wedge s_0)]\right].
$$

**Proof.** From the definition of $G$ it follows by a straightforward calculation that $G(x, A) = \int_0^{\infty} E_x\left[1_A[X(T_1 \wedge t)]\right]e^{-t}dt$, showing the result.

**Lemma 7.2.** For any $x \in E$ and for all $n \in \mathbb{N}$, we have

$$
\begin{align*}
(7.1) \quad & P_x(\tau_n \leq T_n) = 1, \\
(7.2) \quad & P_x(\tau_n < \tau_{n+1}) = 1, \\
(7.3) \quad & P_x\left(\sum_{j=n}^{\infty} 1_{\{\tau_j = T_n\}} = 1\right) = 1.
\end{align*}
$$

For all $n \in \mathbb{N}^*$, $k < n$, $j \in \{k, k+1, \ldots, n-1\}$, and $x \in E$,

$$
(7.4) \quad \tau_n = T_k + \sum_{i=j+1}^{n} s_i,
$$

$P_x$-a.s. on $\{T_k \leq \tau_n < T_{k+1}\} \cap \{\tau_j = T_k\}$. 

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Proposition 7.3. For all $n \in \mathbb{N}_+$, $k < n$,

\[
(7.5) \quad \tau_n 1_{\{T_k \leq \tau_n < T_{k+1}\}} = \sum_{j=k}^{n-1} 1_{\{\tau_j = T_k\}} \left[ T_k + \sum_{i=j+1}^{n} s_i \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}} + 1_{\{T_k = \tau_n\}} T_k.
\]

Proof. From (7.3), we have

\[
(7.6) \quad \tau_n 1_{\{T_k \leq \tau_n < T_{k+1}\}} = \tau_n 1_{\{T_k \leq \tau_n < T_{k+1}\}} \left[ \sum_{j=k}^{n} 1_{\{\tau_j = T_k\}} + \sum_{j=n+1}^{\infty} 1_{\{\tau_j = T_k\}} \right],
\]

and from (7.2), we get $\{T_k \leq \tau_n < T_{k+1}\} \cap \{\tau_j = T_k\} = \emptyset$ for $j \geq n+1$. Consequently, the last term in (7.6) cancels out, and, by using (7.4), the result follows. \( \square \)

Lemma 7.4. For all $(k, p, n) \in \mathbb{N}_+^3$ such that $p \leq n$ and $k < n$, there exists a measurable function $H_{k,p}$ such that

\[
\tau_p 1_{\{T_k \leq \tau_n < T_{k+1}\}} = H_{k,p}(T_1, \ldots, T_k, s_1, \ldots, s_p) 1_{\{T_k \leq \tau_n < T_{k+1}\}}.
\]

Proof. For $p \leq n$ and $k < n$ and from the definition of $\tau_p$, we have

\[
(7.7) \quad \tau_p 1_{\{T_k \leq \tau_n < T_{k+1}\}} = \sum_{j=0}^{p-1} 1_{\{T_j \leq \tau_{p-1} < T_{j+1}\}} \left[ (\tau_{p-1} + s_p) \wedge T_{j+1} \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}}.
\]

However, if $k < p - 1$, then $\{T_j \leq \tau_{p-1} < T_{j+1}\} \cap \{T_k \leq \tau_n < T_{k+1}\} = \emptyset$ for $j \in \{k+1, \ldots, p-1\}$ by using (7.2). Consequently, it follows from (7.7) that

\[
\tau_p 1_{\{T_k \leq \tau_n < T_{k+1}\}} = \sum_{j=0}^{k/(p-1)} 1_{\{T_j \leq \tau_{p-1} < T_{j+1}\}} \left[ (\tau_{p-1} + s_p) \wedge T_{j+1} \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}}
\]

\[
= \sum_{j=0}^{(k-1)/(p-2)} 1_{\{T_j \leq \tau_{p-1} < T_{j+1}\}} \left[ (\tau_{p-1} + s_p) \wedge T_{j+1} \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}}
\]

\[
+ 1_{\{p-1 < k\}} 1_{\{T_{p-1} \leq \tau_{p-1} < T_p\}} \left[ (\tau_{p-1} + s_p) \wedge T_p \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}}
\]

\[
(7.8) \quad + 1_{\{p-1 \geq k\}} 1_{\{T_k \leq \tau_{p-1} < T_{k+1}\}} \left[ (\tau_{p-1} + s_p) \wedge T_{k+1} \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}}.
\]

However, by using (7.1) we have $\{T_{p-1} \leq \tau_{p-1} < T_p\} = \{T_{p-1} = \tau_{p-1}\}$, and $\{T_k \leq \tau_{p-1} < T_{k+1}\} \cap \{T_k \leq \tau_n < T_{k+1}\} = \{T_k \leq \tau_{p-1} \} \cap \{T_k \leq \tau_n < T_{k+1}\}$. On the set $\{T_k \leq \tau_{p-1} < T_{k+1}\} \cap \{T_k \leq \tau_n < T_{k+1}\}$, we get $(\tau_{p-1} + s_p) \wedge T_{k+1} = \tau_{p-1} + s_p$. Taking these remarks into consideration, (7.8) becomes

\[
\tau_p 1_{\{T_k \leq \tau_n < T_{k+1}\}} = \sum_{j=0}^{(k-1)/(p-2)} 1_{\{T_j \leq \tau_{p-1} < T_{j+1}\}} \left[ (\tau_{p-1} + s_p) \wedge T_{j+1} \right]
\]

\[
+ 1_{\{p-1 < k\}} 1_{\{T_{p-1} = \tau_{p-1}\}} \left[ (\tau_{p-1} + s_p) \wedge T_{p} \right]
\]

\[
+ 1_{\{p-1 \geq k\}} 1_{\{T_k \leq \tau_{p-1} < T_{k+1}\}} (\tau_{p-1} + s_p) 1_{\{T_k \leq \tau_n < T_{k+1}\}}.
\]
Consequently, there exists a measurable function $H_{p,k}$ such that

$$\tau_p 1_{\{T_k \leq \tau_n < T_{k+1}\}} = H_{p,k}(\tau_{p-1}, T_1, \ldots, T_k, s_1, \ldots, s_p) 1_{\{T_k \leq \tau_n < T_{k+1}\}}.$$  

Now repeating the same arguments for $\tau_{p-1} 1_{\{T_k \leq \tau_n < T_{k+1}\}}$, the result follows by induction. □

**Lemma 7.5.** Define $G_n = \sigma\{X(t \wedge \tau_n) : t \in \mathbb{R}_+\} \vee \sigma\{\tau_j : j \leq n\}$, and $S_n = \sigma\{s_j : j \leq n\}$. For $n \in \mathbb{N}_*$ and $k < n$, we have that

$$\left[G_n \cap \{T_k \leq \tau_n < T_{k+1}\}\right] \subset \left[\left(S_n \vee \sigma\{X(t \wedge T_k) : t \in \mathbb{R}_+\}\right) \cap \{T_k \leq \tau_n < T_{k+1}\}\right].$$

**Proof.** For all $t \in \mathbb{R}_+$, it follows from Lemma 7.4 that for all $p \leq n$,

$$\{\tau_p \leq t\} \cap \{T_k \leq \tau_n < T_{k+1}\} \in \left(S_n \vee \sigma\{X(t \wedge T_k) : t \in \mathbb{R}_+\}\right) \cap \{T_k \leq \tau_n < T_{k+1}\}.$$  

Now, from Proposition 7.3, we have on the set $\{t \geq T_k\} \cap \{T_k \leq \tau_n < T_{k+1}\}$

$$X(t \wedge \tau_n) = \sum_{j=k}^{n-1} 1_{\{\tau_j = T_k\}} \Phi\left(X(T_k), (t - T_k) \wedge \sum_{i=j+1}^{n} s_i\right) + 1_{\{\tau_n = T_k\}} \Phi(X(T_k), 0).$$

Therefore, for any $B \in \mathcal{B}(\mathbb{R})$ and for all $t \in \mathbb{R}_+$, we have

$$\{X(t \wedge \tau_n) \in B\} \cap \{T_k \leq \tau_n < T_{k+1}\} = \left[\{t < T_k\} \cap \{X(t \wedge T_k) \in B\}\right] \cup \left[\bigcup_{j=k}^{n-1} \{\tau_j = T_k\} \cap \left\{\Phi\left(X(T_k), (t - T_k) \wedge \sum_{i=j+1}^{n} s_i\right) \in B\right\} \cap \{T_k \leq t\}\right] \cup \left[\{\tau_n = T_k\} \cap \{X(T_k) \in B\} \cap \{T_k \leq t\}\right] \cap \{T_k \leq \tau_n < T_{k+1}\}.$$

Using again Lemma 7.4, it follows that

$$\{X(t \wedge \tau_n) \in B\} \cap \{T_k \leq \tau_n < T_{k+1}\} \in \left(S_n \vee \sigma\{X(t \wedge T_k) : t \geq 0\}\right) \cap \{T_k \leq \tau_n < T_{k+1}\},$$

showing the result. □

**Proposition 7.6.** Assume that $\bigvee_{t \geq 0} F_t^x$ and $\sigma\{s_k : k \geq 0\}$ are independent. For $n \in \mathbb{N}_*$ and $k \leq n$, we have that

$$E_x\left[1_{\{t + \tau_n < T_{k+1}\}} 1_{\{T_k \leq \tau_n < T_{k+1}\}} \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}} = F(t, \Theta_n) 1_{\{T_k \leq \tau_n < T_{k+1}\}};$$

where $F(t, x) = 1_{\{t < t(x)\}} e^{-\int_0^t \lambda(x(s)) ds}$.  

**Proof.** Denote $S_{k+1} = T_{k+1} - T_k$. Consider first the case where $n \in \mathbb{N}_*$ and $k = n$. From (7.1) and by Theorem 25.5 in [6] and its proof, it follows that

$$E_x\left[1_{\{t + \tau_n < T_{k+1}\}} 1_{\{T_k \leq \tau_n < T_{k+1}\}} \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}} = F(t, X(\tau_n)) 1_{\{T_k \leq \tau_n < T_{k+1}\}}.$$
Now consider the case where $n \in \mathbb{N}_*$ and $k < n$. From Proposition 7.3, we have

$$E_x \left[ 1_{\{t+\tau_n < T_{k+1}\}} | \mathcal{G}_n \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}} = \sum_{j=k}^{n-1} E_x \left[ 1_{\{\tau_j = T_k\}} 1_{\{S_{k+1} > \sum_{i=j+1}^{n} s_i + t\}} | \mathcal{G}_n \right]$$

(7.9) 

$$+ E_x \left[ 1_{\{\tau_n = T_k\}} 1_{\{S_{k+1} > t\}} | \mathcal{G}_n \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}}.$$

From Lemma 7.5, we obtain that for any $A \in \mathcal{G}_n$, there exists $B \in \mathcal{F}_n \vee \sigma \{X(t \wedge T_k) : t \in \mathbb{R}_+\}$ such that

$$A \cap \{T_k \leq \tau_n < T_{k+1}\} = B \cap \{T_k \leq \tau_n < T_{k+1}\}.$$

By denoting $\mathcal{H}_k = \sigma \{s_j : j \geq 0\} \vee \sigma \{X(t \wedge T_k) : t \in \mathbb{R}_+\}$, we have for $j \leq n - 1$

$$\int_A 1_{\{S_{k+1} \geq \sum_{i=j+1}^{n} s_i + t\}} 1_{\{T_k \leq \tau_n < T_{k+1}\}} 1_{\{\tau_j = T_k\}} dP$$

$$= \int_B 1_{\{\tau_j = T_k\}} E_x \left[ 1_{\{S_{k+1} \geq \sum_{i=j+1}^{n} s_i + t\}} | \mathcal{H}_k \right] dP.$$

However, since $\mathbb{V}_{t \geq 0} \mathcal{F}_t^X$ and $\sigma \{s_k : k \geq 0\}$ are independent, we have that

$$E_x \left[ 1_{\{S_{k+1} \geq \sum_{i=j+1}^{n} s_i + t\}} | \mathcal{H}_{n,k} \right] = F \left( \sum_{i=j+1}^{n} s_i + t, X(T_k) \right).$$

Therefore, by using the semigroup property of $\Phi$, we obtain

$$\int_A 1_{\{S_{k+1} \geq \sum_{i=j+1}^{n} s_i + t\}} 1_{\{T_k \leq \tau_n < T_{k+1}\}} 1_{\{\tau_j = T_k\}} dP$$

$$= \int_A 1_{\{T_k \leq \tau_n < T_{k+1}\}} 1_{\{\tau_j = T_k\}} F(t, X(\tau_n)) dP.$$

Consequently, it follows that for $j \leq n - 1$

(7.10) 

$$E_x \left[ 1_{\{S_{k+1} \geq \sum_{i=j+1}^{n} s_i + t\}} | \mathcal{G}_n \right] = F(t, X(\tau_n))$$

on the set $\{\tau_j = T_k\} \cap \{T_k \leq \tau_n < T_{k+1}\}$. Similarly, we have that

(7.11) 

$$1_{\{\tau_n = T_k\}} E_x \left[ 1_{\{S_{k+1} > t\}} | \mathcal{G}_n \right] = 1_{\{\tau_n = T_k\}} F(t, X(\tau_n)).$$

Combining (7.9), (7.10), and (7.11), we have that

$$E_x \left[ 1_{\{t+\tau_n < T_{k+1}\}} | \mathcal{G}_n \right] 1_{\{T_k \leq \tau_n < T_{k+1}\}} = 1_{\{T_k \leq \tau_n < T_{k+1}\}} F(t, X(\tau_n)) \sum_{j=k}^{n} 1_{\{\tau_j = T_k\}},$$

showing the result.  \( \square \)

Using the previous results, we now present the proof of Theorem 3.1.

**Proof.** Consider $n \in \mathbb{N}_*$ and denote $\sigma \{X(\tau_k) : k \leq n\}$ by $\mathcal{F}_n^\Theta$. From the definition of $\tau_n$ (see (3.1)), we have

$$E_x \left[ 1_A(\Theta_{n+1}) | \mathcal{F}_n^\Theta \right] = E_x \left[ \sum_{k=0}^{n} 1_{\{T_k \leq \tau_n < T_{k+1}\}} 1_A(\{X(\tau_n + s_{n+1} \wedge T_{k+1})\}) | \mathcal{F}_n^\Theta \right].$$
However, by using the facts that $\forall t \geq 0 \mathcal{F}_t^X$ and $\sigma \{ s_k : k \geq 0 \}$ are independent and that \{s_n\}_{n \geq 0} is a sequence of independent and identically distributed $\mathbb{R}_+$-valued random variables with exponential distribution, we obtain

$$E_x \left[ 1_A(\Theta_{n+1}) | \mathcal{F}_n^\Theta \right] = \int_0^{+\infty} E_x \left[ \sum_{k=0}^{n} 1_{\{ T_k \leq \tau_n < T_{k+1} \}} 1_A[X(\tau_n + t) \wedge T_{k+1}] | \mathcal{F}_n^\Theta \right] e^{-t}dt \quad (7.12)$$

where for the last equality we have used $\{ T_k \leq \tau_n < T_{k+1} \} \in \mathcal{G}_n$ and $\mathcal{F}_n^\Theta \subset \mathcal{G}_n$. For $k \leq n$, by using Proposition 7.6 we have that on the set $\{ T_k \leq \tau_n < T_{k+1} \}$

$$E_x \left[ 1_A[X(\tau_n + t) \wedge T_{k+1}] | \mathcal{G}_n \right] = E_{X(\tau_n)} \left[ 1_A[X(1 \wedge t)] \right]. \quad (7.13)$$

Combining (7.12) and (7.13), it follows that

$$E_x \left[ 1_A(\Theta_{n+1}) | \mathcal{F}_n^\Theta \right] = \int_0^{+\infty} E_x \left[ \sum_{k=0}^{n} 1_{\{ T_k \leq \tau_n < T_{k+1} \}} E_{X(\tau_n)} \left[ 1_A[X(1 \wedge t)] \right] | \mathcal{F}_n^\Theta \right] e^{-t}dt$$

$$= \int_0^{+\infty} E_{X(\tau_n)} \left[ 1_A[X(1 \wedge t)] \right] E_x \left[ \sum_{k=0}^{n} 1_{\{ T_k \leq \tau_n < T_{k+1} \}} | \mathcal{F}_n^\Theta \right] e^{-t}dt$$

$$= G(X(\tau_n), A),$$

showing the result. \[ \square \]

8. Proof of Theorem 3.3. In this section we present the proof of Theorem 3.3. Propositions 8.1 and 8.2 show how we can represent $L^R$ as an infinite sum of substochastic kernels. Proposition 8.3 presents iterative equations for $L^R$ and $L^G$, which, combined with the previous propositions and some limit convergence proved in Proposition 8.4, yield the proof of Theorem 3.3.

**Proposition 8.1.** Let $f$ be an $\mathbb{R}_+$-valued measurable function defined on $E$. For $B \in \mathcal{B}(E)$ and $n \in \mathbb{N}$,

$$\quad (8.1) \quad (HI_B)^{n+1} f(x) = E_x \left[ \int_0^{T_1} v_n,B(x, s)e^{-s}f(X(s))1_B(X(s))ds \right],$$

$$\quad (8.2) \quad (HI_B)^n f(x) = E_x \left[ v_n,B(x, T_1)e^{-T_1}f(X(T_1)) \right],$$

where

$$\quad (8.3) \quad v_n,B(x, s) = \frac{1}{n!} \left[ \int_0^s 1_B(\Phi(x, u))du \right]^n.$$

**Proof.** Notice that for $s \geq t$ we have that

$$v_n,B(\Phi(x, t), s-t) = \frac{1}{n!} \left[ \int_0^{s-t} 1_B(\Phi(x, u + t))du \right]^n = \frac{1}{n!} \left[ \int_t^s 1_B(\Phi(x, u))du \right]^n$$

and thus

$$\quad (8.4) \quad \frac{dv_n,B(\Phi(x, t), s-t)}{dt} = -v_{n-1,B}(\Phi(x, t), s-t)1_B(\Phi(x, t)).$$
Let us first show (8.1) by induction on $n$. For $n = 1$ the result follows from the fact that

(8.5) \[ H(x, A) = E_x \left[ \int_0^{T_1} e^{-s} 1_A(X(s)) ds \right]. \]

Suppose it holds for $n$. Note that for any $\mathbb{R}_+^*$-valued measurable function $h$ defined on $\mathbb{R}$, we have

(8.6) \[ E_x \left[ \int_0^{T_1} h(s) ds \right] = \int_0^{t_*(x)} h(s) e^{-\Lambda(x,s)} ds. \]

Combining (8.5) and (8.6), we have

(8.7) \[ (HI_B)^{n+1} f(x) = HI_B(HI_B)^n f(x) = \int_0^{t_*(x)} 1_B(\Phi(x,s))(HI_B)^n f(\Phi(x,s)) e^{-\{s+\Lambda(x,s)\}} ds. \]

However, from the induction hypothesis and (8.6), we have

(8.8) \[ (HI_B)^n f(x) = \int_0^{t_*(x)} v_{n-1,B}(x,u) 1_B(\Phi(x,u)) f(\Phi(x,u)) e^{-(u+\Lambda(x,u))} du. \]

Using the semigroup property of $\Phi$, it follows that

(8.9) \[ (HI_B)^n f(\Phi(x,s)) = e^{\{s+\Lambda(x,s)\}} \int_s^{t_*(x)} v_{n-1,B}(\Phi(x,s), u-s) 1_B(\Phi(x,u)) \times f(\Phi(x,u)) e^{-\{u+\Lambda(x,u)\}} du. \]

Combining (8.7) and (8.8), we obtain that

(8.10) \[ (HI_B)^{n+1} f(x) = \int_0^{t_*(x)} 1_B(\Phi(x,u)) f(\Phi(x,u)) e^{-\{u+\Lambda(x,u)\}} \times \int_0^{t_*(x)} v_{n-1,B}(\Phi(x,s), u-s) 1_B(\Phi(x,s)) 1_{[u,u]}(s) ds du. \]

Now from (8.4), we have

\[ \int_0^u v_{n-1,B}(\Phi(x,s), u-s) 1_B(\Phi(x,s)) ds = v_{n,B}(x,u), \]

and so

(8.11) \[ (HI_B)^{n+1} f(x) = \int_0^{t_*(x)} v_{n,B}(x,u) 1_B(\Phi(x,u)) f(\Phi(x,u)) e^{-\{u+\Lambda(x,u)\}} du. \]

Finally, using (8.6), we obtain

(8.12) \[ (HI_B)^{n+1} f(x) = E_x \left[ \int_0^{T_1} v_{n,B}(x,s) e^{-s} f(X(s)) 1_B(X(s)) ds \right], \]

showing the first part of the result.
The second part of the result can be obtained by using the same arguments and the fact that

\[
E_x [g(T_1) f(X(T_1))] = \int_0^{t_*(x)} \lambda(\Phi(x, s)) e^{-\Lambda(x, s)} g(s) Q_f(\Phi(x, s)) ds \\
+ e^{-\Lambda(x, t_*(x))} g(t_*(x)) Q_f(\Phi(x, t_*(x)))
\]

for any \( R \)-valued measurable function \( f \) (respectively, \( g \)) defined on \( R \) (respectively, \( E \)). Indeed, for \( n = 0 \) the result follows from the fact that

\[ J(x, A) = E_x [e^{-T_1} \mathbf{1}_A(X(T_1))]. \]

Now, suppose it holds for \( n \). Then

\[
(HI_B)^{n+1} J f(x) = \int_0^{t_*(x)} \mathbf{1}_B(\Phi(x, s)) (HI_B)^n J f(\Phi(x, s)) e^{-\{s+\Lambda(x, s)\}} ds
\]

and

\[
(HI_B)^n J f(\Phi(x, t)) = \left[ e^{-\Lambda(x, t_*(x))} Q_f(\Phi(x, t_*(x))) v_{n,B}(\Phi(x, t), t_*(x) - t) \\
+ \int_t^{t_*(x)} \lambda(\Phi(x, s)) e^{-\Lambda(x, s)} Q_f(\Phi(x, s)) v_{n,B}(\Phi(x, t), s - t) ds \right] e^{\{t+\Lambda(x, t)\}}.
\]

Combining the two previous equations and (8.4) gives

\[
(HI_B)^{n+1} J f(x) = \int_0^{t_*(x)} \lambda(\Phi(x, s)) e^{-\{s+\Lambda(x, s)\}} Q_f(\Phi(x, s)) v_{n+1,B}(x, s) ds \\
+ e^{-\{t_*(x)+\Lambda(x, t_*(x))\}} Q_f(\Phi(x, t_*(x))) v_{n+1,B}(x, t_*(x)).
\]

From (8.9) we have \((HI_B)^{n+1} J f(x) = E_x [v_{n,B}(x, T_1) e^{-T_1} f(X(T_1))]\), completing the proof.

**Proposition 8.2.** For any \( A \in \mathcal{B}(E) \) define the kernel \( \mathbb{H}_A \) on \((E, \mathcal{B}(E))\)

\[
\mathbb{H}_A = \sum_{n=0}^{\infty} (HI_A^*)^n.
\]

Then

\[
L^R(x, A) = \sum_{n=0}^{\infty} (\mathbb{H}_A J)^n \mathbb{H}_A \mathbf{1}_A(x).
\]

**Proof.** From Proposition 8.1 we have that for any \( R \)-valued measurable function \( g \) defined on \( E \)

\[
\mathbb{H}_A g(x) = g(x) + E_x \left[ \int_0^{T_1} \sum_{n=0}^{\infty} v_{n,A^c}(x, s) e^{-s} g(X(s)) \mathbf{1}_{A^c}(X(s)) ds \right] \\
= g(x) + E_x \left[ \int_0^{T_1} e^{-s+\int_0^s 1_{A^c}(\Phi(x, u)) du} g(\Phi(x, s)) \mathbf{1}_{A^c}(\Phi(x, s)) ds \right] \\
= g(x) + \int_0^{t_*(x)} e^{-\int_0^t 1_{A^c}(\Phi(x, u)) du} g(\Phi(x, s)) \mathbf{1}_{A^c}(\Phi(x, s)) e^{-\Lambda(x, s)} ds.
\]
However, \( H1_{A}(\Phi(x,s)) = \int_{t}^{\Phi(x,s)} 1_{A}(\Phi(x,s))e^{-[x+\Lambda(x,s)]ds} e^{t+\Lambda(x,t)} \). Consequently,

\[
\mathbb{H}_{A}H1_{A}(x) = \int_{0}^{t_{1}(x)} 1_{A}(\Phi(x,s))e^{-[x+\Lambda(x,s)]} \int_{0}^{s} e^{-f_{0}^{t_{0}} 1_{A}(\Phi(x,u))du} 1_{A^{c}}(\Phi(x,t))dt ds \\
+ H1_{A}(x)
\]

\[
= \int_{0}^{t_{1}(x)} 1_{A}(\Phi(x,s))e^{-[x+\Lambda(x,s)]} \left[ e^{-f_{0}^{t_{0}} 1_{A}(\Phi(x,u))du} - 1 \right] ds + H1_{A}(x)
\]

\[
= \int_{0}^{t_{1}(x)} 1_{A}(\Phi(x,s))e^{-\Lambda(x,s)}e^{-f_{0}^{t_{0}} 1_{A}(\Phi(x,u))du} ds
\]

(8.10)

\[
E_{x} \left[ \int_{0}^{T_{1}} e^{-f_{0}^{t_{0}} 1_{A}(X(u))du} 1_{A}(X(t))dt \right].
\]

Using Proposition 8.1, we have that for any \( \mathbb{R}_{+} \)-valued measurable function \( f \) defined on \( E \)

\[
\mathbb{H}_{A}Jf(x) = E_{x} \left[ \sum_{n=0}^{\infty} v_{n,A^{c}}(x,T_{1})e^{-T_{1}f(X(T_{1}))} \right]
\]

\[
= E_{x} \left[ e^{-T_{1} + \int_{0}^{T_{1}} 1_{A^{c}}(\Phi(x,u))du} f(X(T_{1})) \right]
\]

(8.11)

\[
(\mathbb{H}_{A}J)^{n}\mathbb{H}_{A}H1_{A}(x) = E_{x} \left[ \int_{T_{n}}^{T_{n+1}} e^{-f_{0}^{t_{0}} 1_{A}(X(u))du} 1_{A}(X(t))dt \right].
\]

Taking the sum over \( n \) in (8.12) and recalling that a.s. \( T_{n} \rightarrow \infty \) as \( n \rightarrow \infty \), it follows that

\[
\sum_{n=0}^{\infty} (\mathbb{H}_{A}J)^{n}\mathbb{H}_{A}H1_{A}(x) = E_{x} \left[ \int_{0}^{\infty} e^{-f_{0}^{t_{0}} 1_{A}(X(u))du} 1_{A}(X(t))dt \right] = 1 - E_{x} \left[ e^{-f_{0}^{T_{1}} 1_{A}(X(u))du} \right] = 1 - E_{x} \left[ e^{-\eta_{A}} \right] = LR(x, A),
\]

where the last equality follows from, for instance, Theorem 2.3(ii) of [11].

**Proposition 8.3.** We have that

(8.13) \( LR(x, A) = \mathbb{H}_{A}H(x, A) + \mathbb{H}_{A}JL^{R}(x, A) \),

(8.14) \( L^{G}(x, A) = \mathbb{H}_{A}G(x, A) + \mathbb{H}_{A}JL^{G}(x, A) \).

**Proof.** Note that \( LR(x, A) = \sum_{n=1}^{\infty} (RI_{A^{c}})^{(n-1)}R(x, A) \). Since \( R = \sum_{j=0}^{\infty} J^{j}H \) we have that \( R = H + JR \). Therefore, from [17, Lemma 4.1], it follows that for any \( n \in \mathbb{N}^{*} \)

\[
(RI_{A^{c}})^{n} = (HI_{A^{c}} + JR_{A^{c}})^{n}
\]

\[
= (HI_{A^{c}})^{n} + \sum_{j=1}^{n} (HI_{A^{c}})^{j-1}J(RI_{A^{c}})(RI_{A^{c}})^{n-j}
\]

\[
= (HI_{A^{c}})^{n} + \sum_{j=1}^{n} (HI_{A^{c}})^{j-1}J(RI_{A^{c}})^{n+1-j}.
\]
Taking the sum over \( n \) and using the fact that \( R = H + JR \), we get that

\[
L^R(x, A) = \sum_{n=0}^{\infty} (HI_{A^e})^n R(x, A) + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (HI_{A^e})^{j-1} J (RI_{A^e})^{n+1-j} R(x, A) I_{\{j \leq n\}}
\]

\[
= \mathbb{H}_A R(x, A) + \sum_{j=1}^{\infty} (HI_{A^e})^{j-1} J \sum_{n=j}^{\infty} (RI_{A^e})^{n+1-j} R(x, A)
\]

\[
= \mathbb{H}_A H(x, A) + \mathbb{H}_A JR(x, A) + \mathbb{H}_A J \sum_{n=1}^{\infty} (RI_{A^e})^n R(x, A),
\]

giving (8.13). Similarly for \( G \), we have \( L^G(x, A) = \sum_{n=1}^{\infty} (GI_{A^e})^{n-1} G(x, A) \) and so

\[
L^G(x, A) = \sum_{n=0}^{\infty} (HI_{A^e})^n G(x, A) + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (HI_{A^e})^{j-1} JI_{A^e} (GI_{A^e})^{n-j} G(x, A) I_{\{j \leq n\}}
\]

\[
= \mathbb{H}_A G(x, A) + \sum_{j=1}^{\infty} (HI_{A^e})^{j-1} JI_{A^e} \sum_{n=j}^{\infty} (GI_{A^e})^{n-j} GI_A(x),
\]

showing the last part of the result.

**Proposition 8.4.** For every \( x \in E \) and for \( A \in \mathcal{B}(E) \), we have that

\[
\lim_{n \to \infty} (\mathbb{H}_A J)^n L^R(x, A) = 0
\]

and the limit

\[
\lim_{n \to \infty} (\mathbb{H}_A J I_{A^e})^n L^G(x, A)
\]

exists.

**Proof.** From (8.13) we have that for every \( x \in E \) and all \( n \in \mathbb{N}^* \),

\[
\sum_{k=0}^{n} (\mathbb{H}_A J)^k \mathbb{H}_A H1_A(x) + (\mathbb{H}_A J)^{n+1} G^R(x, A) = L^R(x, A).
\]

Taking the limit as \( n \to \infty \) in (8.15), the first part of the result follows from (8.9).

Similarly, from (8.14) we have that every \( x \in E \) and all \( n \in \mathbb{N}^* \)

\[
\sum_{k=0}^{n} (\mathbb{H}_A J I_{A^e})^k \mathbb{H}_A G1_A(x) + (\mathbb{H}_A J I_{A^e})^{n+1} L^G(x, A) = L^G(x, A).
\]

Since \( \sum_{k=0}^{n} (\mathbb{H}_A J I_{A^e})^k \mathbb{H}_A G1_A(x) \geq 0 \), and \( (\mathbb{H}_A J I_{A^e})^{n+1} L^G(x, A) \geq 0 \) for all \( x \in E \) and all \( n \in \mathbb{N}^* \), it follows that \( \lim_{n \to \infty} \sum_{k=0}^{n} (\mathbb{H}_A J I_{A^e})^k \mathbb{H}_A G1_A(x) \) exists in \( \mathbb{R}_+ \), implying the existence of \( \lim_{n \to \infty} (\mathbb{H}_A J I_{A^e})^n L^G(x, A) \), completing the proof.

Using the previous results, we now present the proof of Theorem 3.3.

**Proof.** From (8.13) and (8.14), it follows that for all \( x \in E \),

\[
L^G(x, A) - L^R(x, A) = \mathbb{H}_A (G(x, A) - H(x, A)) + \mathbb{H}_A J I_{A^e} U^G(x, A)
\]

\[
- \mathbb{H}_A J (I_{A^e} + I_A) L^R(x, A)
\]

\[
= \mathbb{H}_A J (x, A) - \mathbb{H}_A J I_{A^e} L^R(x, A)
\]

\[
+ \mathbb{H}_A J I_{A^e} [L^G(x, A) - L^R(x, A)]
\]

\[
= \mathbb{H}_A J I_{A^e} [1_E(x) - L^R(x, A)]
\]

\[
+ \mathbb{H}_A J I_{A^e} [L^G(x, A) - L^R(x, A)].
\]
Consequently, for all \( x \in E \) and all \( n \in \mathbb{N}^* \),

\[
L^G(x, A) - L^R(x, A) = \sum_{k=0}^{n} [\mathbb{H}_A J I_{A^c}]^n \mathbb{H}_A J I_A [1_E(x) - L^R(x, A)] + [\mathbb{H}_A J I_{A^c}]^{n+1} [L^G(x, A) - L^R(x, A)].
\]

From Proposition 8.4, and using the fact that \( \lim_{n \to \infty} (\mathbb{H}_A J I_{A^c})^n L^G(x, A) \) is non-negative, it follows that

\[
L^G(x, A) - L^R(x, A) \geq \sum_{k=0}^{\infty} [\mathbb{H}_A J I_{A^c}]^n \mathbb{H}_A J I_A [1_E(x) - L^R(x, A)].
\]

Since for all \( x \in E, A \in B(E), L^R(x, A) \leq 1_E(x) \), the result follows. \( \square \)

**Acknowledgment.** The authors would like to express their gratitude to the associate editor and referees for their suggestions and helpful comments.

**REFERENCES**


