POSITIVITY PROPERTIES OF THE FOURIER TRANSFORM AND THE STABILITY OF PERIODIC TRAVELLING-WAVE SOLUTIONS

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POSITIVITY PROPERTIES OF THE FOURIER TRANSFORM AND THE STABILITY OF PERIODIC TRAVELLING-WAVE SOLUTIONS*

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Abstract. In this paper we establish a method to obtain the stability of periodic travelling-wave solutions for equations of Korteweg–de Vries-type \( u_t + u^p u_x - M u_x = 0 \), with \( M \) being a general pseudodifferential operator and where \( p \geq 1 \) is an integer. Our approach uses the theory of totally positive operators, the Poisson summation theorem, and the theory of Jacobi elliptic functions. In particular we obtain the stability of a family of periodic travelling waves solutions for the Benjamin–Ono equation. The present technique gives a new way to obtain the existence and stability of cnoidal and dnoidal waves solutions associated with the Korteweg–de Vries and modified Korteweg–de Vries equations, respectively. The theory has prospects for the study of periodic travelling-wave solutions of other partial differential equations.

Key words. dispersive equations, Korteweg–de Vries-type equations, periodic travelling waves, Jacobi elliptic functions, nonlinear stability

AMS subject classifications. 76B25, 35Q51, 35Q53

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1. Introduction. One of the main properties of dispersive nonlinear evolution equations is that usually they sustain steadily translating waves called travelling waves. These solutions imply a balance between the effects of nonlinearity and of frequency dispersion. By depending on specific boundary conditions on the wave’s shape, for instance, in the case of water waves, these special states of motion can arise either solitary or periodic waves. The study of these special steady waveforms is essential to the explanation of many wave phenomena observed in the practice, for instance, in surface water waves propagating in a canal, in propagation of internal waves, or in shallow-water ocean surface waves (see Benjamin [12], [13], [14] and Osborne et al. [44]).

In the water wave context, Constantin in [21] and Constantin and Escher in [22] analyzed a free boundary problem for harmonic functions and showed that periodic or solitary travelling waves possess stability properties within the shallow-water regime (see also Toland [47] and Constantin and Strauss [23] and the citations therein). Moreover, various nonlinear dispersive model equations are an accurate approximation to the governing equations for water waves (see [5]). From these considerations, questions about the stability of travelling waves and their existence as exact solutions of the dynamical equations are very important.

The solitary waves are in general single crested, symmetric, localized travelling waves, whose hyperbolic sech profiles are well known (see Ono [43] and Benjamin [16] for the existence of solitary waves of algebraic type or with a finite number of oscillations). The study of the nonlinear stability or instability in the form of solitary waves has had a terrific development and refinement in recent years. The proofs have
been simplified and sufficient conditions were obtained to ensure the stability to small localized perturbations in the waveform. Those conditions have shown to be effective in a variety of circumstances; see, for example, [2], [3], [4], [14], [17], [27], [28], [49], and [48].

The situation regarding periodic travelling waves is very different. The stability and the existence of explicit formulas of these progressive wavetrains have received comparatively little attention. A first study of these waveforms was determined by Benjamin in [16] with regard to the periodic steady solutions called cnoidal waves, which were found initially by Korteweg and de Vries in [34] for the equation currently called the Korteweg–de Vries equation (KdV henceforth):  

\[ u_t + uu_x + u_{xxx} = 0, \]

where \( u = u(x,t) \) is a real-valued function of the two variables \( x, t \in \mathbb{R} \). Benjamin put forward an approach to the stability of cnoidal waves in the form

\[ \varphi(\xi) = \beta_2 + (\beta_3 - \beta_2)cn^2 \left( \sqrt{\frac{\beta_3 - \beta_1}{12}}, k \right), \]

but did not provide a detailed justification of his assertions, and several aspects seem problematic. The first result of stability for periodic solutions of the KdV was obtained by McKean in [37], who considered the orbital stability of all periodic finite-genus solutions with respect to perturbations of the same period. McKean’s approach was based on the integrable structure of the KdV. More recently Angulo, Bona, and Scialom in [10] returned to Benjamin’s original question and gave a complete theory of stability of cnoidal waves for (1.1) with respect to perturbations of the same period (see also [7]). The approach for obtaining this result was based on the ideas of Bona, Weinstein, and Grillakis, Shatah, and Strauss (see [17], [27], [48]) but adapted to the periodic context. So new theories of stability for other dispersive equations such as the focusing nonlinear Schrödinger equation and the modified KdV has been obtained (see Angulo [9], [8]). It is remarkable to see that in all these works the use of an elaborated spectral theory for the periodic eigenvalue problem was necessary (see [29], [36]),

\[ \begin{cases} 
\frac{d^2}{dx^2} \Psi + [\rho - n(n+1)k^2sn^2(x;k)] \Psi = 0, \\
\Psi(0) = \Psi(2K(k)), \quad \Psi'(0) = \Psi'(2K(k)), 
\end{cases} \]

with specific values of \( n \in \mathbb{N} \). In (1.2), \( sn(\cdot;k) \) represents the Jacobi elliptic function of type snoidal with modulus \( k \), \( k \in (0,1) \), and \( K \) represents the complete elliptic integral of the first kind defined by

\[ K(k) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}. \]

We recall that the second order differential equation in (1.2) is known as the Jacobi form of the Lamé equation.

We note also that Gardner in [26] provided a theory for determining that the large wavelength periodic waves are linearly unstable whenever the limiting homoclinic wave (solitary wave) is unstable. He applied it for diverse types of nonlinear evolution equations in one space variable, in the case of the generalized KdV equations

\[ u_t + uu_x + u_{xxx} = 0, \]
$p \in \mathbb{N}$, and assuming that this equation admits a family of large wavelength periodic waves $U^\alpha$ such that the period $T_\alpha$ tends to infinity as $\alpha$ tends to zero, then they are unstable whenever $p > 4$ and $\alpha > 0$ is sufficiently small.

Then, the main focus in this paper will be the study of the existence and stability of periodic travelling-wave solutions for equations of the form

\begin{equation}
    u_t + u^p u_x - Mu_x = 0, \tag{1.3}
\end{equation}

where $p \geq 1$ is an integer and $M$ is a differential or pseudodifferential operator in the framework of periodic functions. $M$ is defined as a Fourier multiplier operator by

\begin{equation}
    \hat{M}g(k) = \alpha(k)\hat{g}(k), \quad k \in \mathbb{Z}, \tag{1.4}
\end{equation}

where the symbol $\alpha$ of $M$ is assumed to be a measurable, locally bounded, even function on $\mathbb{R}$, satisfying the conditions

\begin{equation}
    A_1|k|^{m_1} \leq \alpha(k) \leq A_2(1 + |k|)^{m_2} \tag{1.5}
\end{equation}

for $m_1 \leq m_2$, $|k| \geq k_0$, $\alpha(k) > b$ for all $k \in \mathbb{Z}$ and $A_i > 0$. The travelling-wave solutions for (1.3) will have the form

\begin{equation}
    u(x, t) = \varphi_c(x - ct), \tag{1.6}
\end{equation}

where the profile $\varphi_c$ is a smooth periodic function with an a priori fundamental period $2L$, $L > 0$. Hence substituting this form of $u$ into (1.3) and integrating once (with the integration constant being considered zero throughout our theory), one obtains that $\varphi = \varphi_c$ is the solution of the equation

\begin{equation}
    (M + c)\varphi - \frac{1}{p + 1}\varphi^{p+1} = 0. \tag{1.6}
\end{equation}

Associated with (1.6) we consider the linear, closed, unbounded, self-adjoint operator $\mathcal{L} : D(\mathcal{L}) \to L^2_{\text{per}}([-L, L])$ defined on a dense subspace of $L^2_{\text{per}}([-L, L])$ by

\begin{equation}
    \mathcal{L}u = (M + c)u - \varphi^pu. \tag{1.7}
\end{equation}

From the theory of compact symmetric operators applied to the periodic eigenvalue problem

\begin{equation}
    \begin{cases}
        \mathcal{L}\psi = \lambda\psi, \\
        \psi(-L) = \psi(L), \quad \psi'(-L) = \psi'(L),
    \end{cases} \tag{1.8}
\end{equation}

it is possible to see that the spectrum of $\mathcal{L}$ is a countable infinite set of eigenvalues, $\{\lambda_n\}$, with

\begin{equation}
    \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots, \tag{1.9}
\end{equation}

where $\lambda_n \to \infty$ as $n \to \infty$ (see Proposition 3.1 below for a proof of this assertion). In particular, from (1.6) we obtain that $\mathcal{L}$ has zero as an eigenvalue with eigenfunction
$d\varphi/dx$. As is well known this property of $\mathcal{L}$ is deduced from the invariance of the solutions of (1.3) by translations.

A set of sharp conditions is available in the literature to imply the stability of the orbit generated by $\varphi_c$, namely, $\Omega_{\varphi_c} = \{ \varphi_c(\cdot + y) : y \in \mathbb{R} \}$. So, we say that $\Omega_{\varphi_c}$ (or $\varphi_c$) is stable in $H^m_{pe\bar{R}}([-L, L])$ by the periodic flow generated by (1.3) if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for $u_0 \in H^m_{pe\bar{R}}([-L, L])$ with $d(u_0, \Omega_{\varphi_c}) \equiv inf_{y \in \mathbb{R}} \|u_0 - \varphi_c(\cdot + y)\|_{H^m_{pe\bar{R}}} < \delta$, the solution $u$ of (1.3) with $u(x, 0) = u_0$ is global in time and satisfies $d(u(t), \Omega_{\varphi_c}) < \epsilon$ for all $t \in \mathbb{R}$. Thus, from [14], [17], [49], [27] the conditions that imply stability are the following:

\begin{equation}
\tag{1.10}
\begin{align*}
(P_0) & \text{ there is a nontrivial smooth curve of periodic solutions for (1.6) of the form } c \in I \subseteq \mathbb{R} \to \varphi_c \in H^m_{pe\bar{R}}([-L, L]); \\
(P_1) & \text{ $\mathcal{L}$ has an unique negative eigenvalue } \lambda, \text{ and it is simple;} \\
(P_2) & \text{ the eigenvalue } 0 \text{ is simple;} \\
(P_3) & \frac{d}{dc} \int_{-L}^{L} \varphi_c^2(x)dx > 0.
\end{align*}
\end{equation}

The problem about the existence of a nontrivial smooth curve of periodic solutions in the form required by $(P_0)$ above presents new and delicate aspects that need to be handled. The possibility of finding explicit solutions for (1.6) will depend naturally on the form of $M$. If it is a differential operator of the form $M = -\partial_x^2$, then we use the quadrature method (it means writing (1.6) in the form $[\varphi_c']^2 = F(\varphi_c)$), and the theory of elliptic functions has shown to be a main tool. So, the solutions will depend on the Jacobi elliptic functions of snoidal, cnoidal, and dnoidal types (see [9], [8], [10], [15]). Now, since the period of these functions depends on the complete elliptic integral $K(k)$, we have that the elliptic modulus $k$ will depend on the velocity $c$, and therefore we have that a priori the period of $\varphi_c$ will depend on $c$. Hence, by using the implicit function theorem, the wanted smooth branch of periodic solutions with a fixed minimal period has been obtained in many cases. We note that the procedure of the quadrature method in general does not work if $M$ is a pseudodifferential operator such as the nonlocal operator $\mathcal{H}\partial_x$, with $\mathcal{H}$ being the Hilbert transform. In this paper we will make a different approach to obtain explicit solutions of (1.6) for a specific form of $M$ and values of $p$. This approach will be based on the classical Poisson summation theorem (see [38], [45], [46]). At least two important advantages of this approach can be obtained: The first one is that it can be used for obtaining solutions when $M$ is a pseudodifferential operator, for example, in the case of $\mathcal{H}\partial_x$. The other one is that related with computing the integral in (1.10). In general obtaining property $(P_3)$ can be very difficult in the periodic case, as the results that have appeared in the literature have shown, since the use of nontrivial identities for the complete elliptic integrals of the first and the second kinds sometimes come on the scene as a fundamental piece in the analysis. As we will see our method to obtain property $(P_3)$ can be very easy as a combination of the Poisson summation theorem and the Parseval theorem.

With regard to conditions $(P_1)$ and $(P_2)$, the problem is very delicate. One of the most remarkable results in the theory of stability of solitary wave solutions was given by Albert [2] and Albert and Bona [3], where sufficient conditions to obtain properties $(P_1)$ and $(P_2)$ were given. The advantage of that approach is that it does not require an explicit computation of the spectrum of the linear operator (1.7), since $(P_1)$ and
(P2) are obtained exclusively from positivity properties of the Fourier transform of the solitary wave in question. The present paper establishes an extension of the theory in [2] and [3] in the case of periodic travelling-wave solutions. The periodic problem has new points not encountered when considering issues related to the solitary waves. Our analysis also relies upon the theory of totally positive operators, and so the class PF(2) defined by Karlin in [32] (see also [2]) is basic in our study.

Our theory leads to a significant simplification of some recent proofs of stability of periodic travelling-wave solutions of KdV-type equations (see [9], [7], [10]) such as in the case of the KdV and the modified KdV equations, since in those cases the verification of properties (P1) and (P2) required the determination of the instability intervals associated with the Lamé equation in (1.2) and of an explicit formula of at least the first three eigenvalues ρ (see [36]). Our analysis does not need this information.

Such as will be shown in section 4, this method establishes the first result about the stability of periodic travelling-wave solutions found by Benjamin in [15] for the Benjamin–Ono equation

\begin{equation}
(1.11)
\quad u_t + uu_x - H u_{xx} = 0,
\end{equation}

where H denotes the periodic Hilbert transform defined by

\begin{equation}
(1.12)
\quad \mathcal{H}f(x) = \frac{1}{2L} \text{p.v.} \int_{-L}^{L} \cotg \left( \frac{\pi(x-y)}{2L} \right) f(y) \, dy.
\end{equation}

The associated periodic waves for (1.11) with a minimal period 2L are given for \( c > \frac{\pi}{L} \) as

\begin{equation}
(1.13)
\quad \varphi_c(x) = \frac{2\pi}{L} \left( \frac{\text{senh}(\gamma)}{\cosh(\gamma) - \cos \left( \frac{\pi x}{2L} \right)} \right),
\end{equation}

such that \( \gamma > 0 \) satisfies \( \tanh(\gamma) = \frac{\pi}{2L} \).

It is important to note that the stability results now presented are obtained by periodic initial disturbances having the same minimal period of our periodic solutions. This method cannot be extended for obtaining stability results with more general periodic perturbations, for instance, by periodic disturbances of two times the minimal period of our periodic solutions. In section 6 we give an explanation of this fact.

The plan of this paper is as follows. The next section is devoted to describing briefly the notation that will be used and making a few preliminary remarks regarding periodic Sobolev spaces, the Poisson summation theorem, and some results of global well-posedness in the periodic case of the KdV, modified KdV, and the Benjamin–Ono equations. Sections 3 and 4 contain our full theory which relates positivity properties of periodic travelling-wave solutions to the stability theory of [27]. Applications of section 4 to specific periodic travelling waves are presented in section 5. In section 6 some comments about the PF(2) property in the periodic case are established. Finally, the appendix contains some basic properties of the elliptic integrals which are relevant to the theory of section 5, and the proof of an inequality of Poincaré–Wintinger type for nonlocal operators is established.

2. Notation and preliminaries.

2.1. Function classes. Let \( \Omega \) be an open set of the real line \( \mathbb{R} \) and \( 1 \leq p \leq \infty \); then \( L^p(\Omega) \) is the usual Banach space of (equivalence classes of) real-
complex-valued Lebesgue measurable functions defined on \( \Omega \) provided with the norm

\[
\|f\|_p = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}}
\]

if \( 1 \leq p < \infty \). When \( p = \infty \) we have \( \|f\|_\infty = \sup_{x \in \Omega} |f(x)| \). When \( p = 2 \) the Banach space \( L^p(\Omega) \) is a Hilbert space with inner product defined by \( (f, g)_2 = \int_{\Omega} f(x)g(x) \, dx \), where \( f, g \in L^2(\Omega) \). The \( L^2 \)-based Sobolev spaces of periodic functions are defined as follows \([30]\). If \( P = C^\infty_{per} \) denotes the collection of all of the functions \( f : \mathbb{R} \to \mathbb{C} \) which are \( C^\infty \) and \( 2 \)-periodic with period \( 2l > 0 \), the collection \( \mathcal{P}' \) of all continuous linear functionals from \( P \) into \( \mathbb{C} \) is the set of periodic distributions. If \( \Psi \in \mathcal{P}' \), we denote the evaluation of \( \Psi \) at \( \varphi \) by \( \langle \Psi, \varphi \rangle = \langle \hat{\Psi}, \Theta_k \rangle \), for \( \varphi \in \mathcal{P} \). For \( k \in \mathbb{Z} \), let \( \Theta_k(x) = e^{ik\pi x}, \, x \in \mathbb{R} \). The Fourier transform of \( \Psi \in \mathcal{P}' \) is a function \( \hat{\Psi} : \mathbb{Z} \to \mathbb{C} \) defined by \( \hat{\Psi}(k) = \frac{1}{2l} \langle \Psi, \Theta_k \rangle, \, k \in \mathbb{Z} \). \( \hat{\Psi}(k) \) are called the Fourier coefficients of \( \Psi \). As usual, a function \( \psi \in L^p([-l, l]) \), \( p \geq 1 \), is an element of \( \mathcal{P}' \) by defining

\[
\langle \psi, \varphi \rangle = \frac{1}{2l} \int_{-l}^{l} \psi(x)\varphi(x) \, dx, \quad \varphi \in \mathcal{P}.
\]

If \( \psi \in L^p([-l, l]) \) for some \( p \geq 1 \), then, for \( k \in \mathbb{Z} \),

\[
\hat{\psi}(k) = \frac{1}{2l} \int_{-l}^{l} \psi(x)e^{-ik\pi x} \, dx.
\]

The Fourier inverse transform of a sequence \( \alpha = (\alpha_k)_{k \in \mathbb{Z}} \in \mathcal{S}(\mathbb{Z}) \), where \( \mathcal{S}(\mathbb{Z}) \) denotes the space of the rapidly decreasing sequences, is the function \( \hat{\alpha} \in \mathcal{P} \) defined by \( \hat{\alpha}(x) = \sum_{k \in \mathbb{Z}} \alpha_k \Theta_k(x) \). We consider the space \( \mathcal{P}' \) provided with the usual weak-star topology, but it will not be needed here. We denote by \( C_{2l} \) the space of the continuous and \( 2l \)-periodic functions. Let \( \alpha = (\alpha_k)_{k \in \mathbb{Z}} \) be a sequence of complex value. The Hilbert space \( \ell^2 := \ell^2(\mathbb{Z}) \) is defined by

\[
\ell^2 = \left\{ \alpha; \, ||\alpha||_{\ell^2} := \left( \sum_{k = -\infty}^{+\infty} |\alpha_k|^2 \right)^{\frac{1}{2}} < \infty \right\}.
\]

For \( s \in \mathbb{R} \), the Sobolev space \( H^s_{per}([-l, l]) := H^s_{2l} \) is the set of all \( f \in \mathcal{P}' \) such that

\[
||f||^2_{H^s_{2l}} = 2l \sum_{k = -\infty}^{+\infty} (1 + |k|^2)^s |\hat{f}(k)|^2 < \infty.
\]

The collection \( H^s_{2l} \) is a Hilbert space with inner product

\[
(f, g)_{H^s_{2l}} = 2l \sum_{k = -\infty}^{+\infty} (1 + |k|^2)^s \hat{f}(k)\overline{\hat{g}(k)}.
\]

When \( s = 0 \), \( H^0_{2l} \) is a Hilbert space that is isometrically isomorphic to a subspace of \( L^2([-l, l]) \) and \( (f, g)_{H^0_{2l}} = (f, g) = \int_{-l}^{l} f(x)\overline{g(x)} \, dx \). The space \( H^0_{2l} \) will be denoted by \( L^2_{2l} \), and its norm will be \( ||\cdot||_{L^2_{2l}} \). Of course, \( H^s_{2l} \subseteq L^2_{2l} \) for all \( s \geq 0 \), and we have
for $s > 1/2$ the Sobolev embedding $H^s_{2t} \hookrightarrow C_{2t}$ (see [30]). The space $\ell^2_{s,2t} := \ell^2_{s,2t}(\mathbb{Z})$, $s \in \mathbb{R}$, is defined by

$$\ell^2_{s,2t}(\mathbb{Z}) := \left\{ \alpha = (\alpha_k)_{k \in \mathbb{Z}}; \| \alpha \|_{\ell^2_s} := \left(2l \sum_{k=-\infty}^{+\infty} (1 + |k|^2)^s |\alpha_k|^2 \right)^{\frac{1}{2}} < +\infty \right\}.$$ 

$\ell^2_{s,2t}$ is a Hilbert space with inner product

$$(\alpha, \beta)_{\ell^2_{s,2t}} = 2l \sum_{k=-\infty}^{+\infty} (1 + |k|^2)^s \alpha_k \overline{\beta_k},$$

where $\alpha = (\alpha_k)_{k \in \mathbb{Z}}$ and $\beta = (\beta_k)_{k \in \mathbb{Z}}$. Then we have that $f \in H^s_{2t}$ if and only if $(\hat{f}(k))_{k \in \mathbb{Z}} \in \ell^2_{s,2t}$, and so from the Parseval theorem (see [30]), $||\hat{f}||^2_{L^2} = \frac{1}{2\pi} ||f||^2_{L^2}$, it follows that $||f||_{H^s_{2t}} = ||\hat{f}||_{\ell^2_{s,2t}}$. The convolution of two sequences $\alpha$ and $\beta$ is the sequence $\alpha \ast \beta$ defined, for all $k \in \mathbb{Z}$, by $(\alpha \ast \beta)_k = \sum_{j=-\infty}^{+\infty} \alpha_{k-j} \beta_j$, whenever the right-hand side of the identity above makes sense. Next, we present some results that we will need throughout this work. We start with the Young inequality (see [38]).

**Proposition 2.1.** Let $\alpha \in \ell^1(\mathbb{Z})$ and $\beta \in \ell^2(\mathbb{Z})$. Then $\alpha \ast \beta \in \ell^2(\mathbb{Z})$. Moreover,

$$||\alpha \ast \beta||_{\ell^2} \leq ||\alpha||_{\ell^1} ||\beta||_{\ell^2}.$$ 

In particular, for every $\alpha \in \ell^1$ fixed, the linear operator $\beta \in \ell^2 \mapsto \alpha \ast \beta \in \ell^2$ is continuous.

Now, we present the Poisson summation theorem. It will be used to find the explicit form of the periodic travelling-wave solutions for some equations.

**Theorem 2.1.** Let $f(x) = \int_{-\infty}^{+\infty} f(y) e^{-ixy} dy$ and $f(y) = \int_{-\infty}^{+\infty} \hat{f}(x) e^{ixy} dx$ satisfying

$$|f(y)| \leq \frac{A}{(1 + |y|)^{1+\delta}} \quad \text{and} \quad |\hat{f}(x)| \leq \frac{A}{(1 + |x|)^{1+\delta}},$$

where $\delta > 0$ and $A > 0$ (then $\hat{f}$ and $f$ can be assumed to be continuous functions). Thus, for $L > 0$,

$$\sum_{n=-\infty}^{+\infty} f(x + 2Ln) = \frac{1}{2L} \sum_{n=-\infty}^{+\infty} \hat{f}\left(\frac{n}{2L}\right) e^{i\frac{2\pi n x}{L}}.$$ 

The two series above converge absolutely.

**Proof.** See [38], [45], and [46].

**2.2. Results of global well-posedness.** Some results about local and global well-posedness associated with (1.3) in the periodic case were initially established in [1]. Here we establish two results that we will use in our theory.

**Theorem 2.2.** Let $s \geq 1$ be given. For each $u_0 \in H^s_{2t}$ there is a unique solution of (1.3), for the cases $p = 1, 2$ and $M = -\partial_x^2$, that for each $T > 0$ lies in $C(0,T;H^s_{2t})$. Moreover, the correspondence $u_0 \mapsto u$ is an analytic function of the relevance function spaces.

**Proof.** See [20].

**Theorem 2.3.** Let $s \geq \frac{1}{2}$ be given. For each $u_0 \in H^s_{2t}$ there is a unique solution of (1.3), for the cases $p = 1$ and $M = \partial_x$, that for each $T > 0$ lies in $C(0,T;H^s_{2t})$. 

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Moreover, the correspondence \( u_0 \mapsto u \) is a continuous function of the relevance function spaces.

Proof. See [39], [41], and [40]. \( \square \)

3. Basic functional spaces. In this section we establish the main spaces in our study of the stability of periodic wave solutions associated with (1) and (7) for the KdV and mKdV cases.

For compact and self-adjoint operators we have the following characterization of the spectrum of
\[ M \]
for all \( s \) where \( M = \mathcal{H} \partial_x \), with the symbol being \( \frac{2}{|n|} \) and \( p = 1 \). Here, \( \mathcal{H} \) denotes the periodic Hilbert transform defined in (1.12) and such that via the Fourier transform satisfies 
\[ \mathcal{H} \hat{f}(n) = -\text{sgn}(n) \hat{f}(n) \] for all \( n \in \mathbb{Z} \). Then, our three periodic travelling-wave equations are
\[
\begin{align*}
\mathcal{H} \varphi_c'' + c \varphi_c - \frac{1}{2} \varphi_c^2 &= 0 \quad \text{(BO),} \\
\varphi_c'' + \frac{1}{2} \varphi_c^2 - c \varphi_c &= 0 \quad \text{(KdV),} \\
\varphi_c'' + \frac{1}{3} \varphi_c^3 - c \varphi_c &= 0 \quad \text{(mKdV).}
\end{align*}
\]

**Remark 3.1.** (a) Here, we shall consider a more convenient form for the mKdV travelling-wave equation, namely,

\[
\varphi_c'' + \varphi_c^3 - c \varphi_c = 0.
\]

(b) The periodic travelling-wave solutions for the KdV and mKdV will be considered of period \( 2L \), but the periodic travelling-wave solution for the BO equation will be considered of period \( 2L \) only by convenience.

(c) After a “bootstrap” argument, we can conclude that every \( \varphi_c \) belongs to \( H^s_{2L} \) for all \( s \in \mathbb{R} \). Thus, \( \varphi \) is infinitely differentiable, with all derivatives in \( L^2_{2L} \).

We will suppose that \( c > -b \), where \( b \) satisfies \( \alpha(k) > b \) for all \( k \in \mathbb{Z} \). With this condition \( M + c \) represents a positive operator. Then, by using the spectral theorem for compact and self-adjoint operators we have the following characterization of the spectrum of \( L \) defined in (1.7).

**Proposition 3.1.** The operator \( L \) in (1.7) is a closed, unbounded, self-adjoint operator on \( L^2_{2L} \), whose spectrum consists of an enumerable (infinite) set of eigenvalues \( \{ \lambda_k \}_{k = 0}^{\infty} \) satisfying \( \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \), and \( \lambda_k \to \infty \) as \( k \to \infty \). In particular, \( L \) has 0 as an eigenvalue with eigenfunction \( \frac{d}{dx} \varphi_c \).

Proof. We suppose that our periodic functions have period \( L \). Clearly \( L \) defined on \( H^s_{2L} \) is a closed, unbounded, self-adjoint operator on \( L^2_{per}([0, L]) \). Let us proof that the spectrum of \( T := M + c \) is a countable infinite set of eigenvalues, \( \{ \gamma_n \} \), with

\[
\gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots,
\]

where \( \gamma_n \to \infty \) as \( n \to \infty \). In fact, let \( R_c = (M + c)^{-1} \), whose symbol is \( \frac{1}{c + \alpha(k)} \) for \( k \in \mathbb{Z} \). Since \( \frac{1}{c + \alpha(k)} \in \ell^2(\mathbb{Z}) \) we have that there is a unique \( G_c \in L^2_{per}([0, L]) \) such that \( G_c(k) = \frac{1}{c + \alpha(k)} \), and, because of this, we have the action

\[
R_c f(x) = \frac{1}{L} \int_0^L G_c(x - y) f(y) dy,
\]

defined for \( f \in L^2_{per}([0, L]) \). Since \([0, L]\) is a bounded set we have that the kernel \( G_c(x, y) := G_c(x - y) \in L^2([0, L] \times [0, L]) \). So, \( R_c \) is a Hilbert–Schmidt operator on
\( L^2_{\text{per}}([0, L]) \) (see [33]), and therefore \( R_c \) is a compact operator on \( L^2_{\text{per}}([0, L]) \) for all \( c > 0 \) (here we supposed without loss of generality that \( b = 0 \)), and so we obtain (3.3).

Next, we will show that there is a \( \mu_1 \) (large enough) such that \( M = (L + \mu_1)^{-1} \) exists and is a bounded, positive, compact, and self-adjoint operator. In fact, first of all, it is easy to see that \( \mathcal{L} \) is limited below; that is, if \( f \in D(\mathcal{L}) \), we have \( \langle Lf, f \rangle \geq -\beta(f, f) \), where \( \beta = ||\varphi_c||_{L^2_{\text{per}}}^2 + c \). Then, we can choose a \( \mu_1 \) such that \( L + \mu_1 > 0 \); that is, \( M \) is positive. We denote \( \mu_1 := \mu \) only by convenience. Let \( \nu \) be a positive number such that \( \nu + \varphi_c - c > 0 \) and \( \nu + \mu > 0 \). Thus, for \( \mu > 0 \) we have \( f = (\mathcal{L} + \mu)g \Leftrightarrow (I - \mathcal{L})g = \mathcal{Y} f \), where \( M = R_{\mu + \nu}[(\nu + \varphi_c - c)g], \mathcal{Y} = R_{\nu + \mu} \), and we denote \( h = \nu + \varphi_c - c \). Next, from the Parseval theorem, it follows that

\[
||Mg||_{L^2_{\text{per}}} \leq \sup_{k \in \mathbb{Z}} \left\{ \frac{1}{\alpha(k) + \nu + \mu} \right\} ||h||_{L^2_{\text{per}}} ||g||_{L^2_{\text{per}}}.
\]

Thus, we can choose \( \mu \) such that \( ||M||_{B(L^2_{\text{per}})} < 1 \) and \( \mathcal{L} + \mu > 0 \). Then, \( I - \mathcal{L} \) is invertible, and we have \( g = (I - \mathcal{L})^{-1} \mathcal{Y} f \) and write \( M = (L + \mu)^{-1} = (I - \mathcal{L})^{-1} \mathcal{Y} \), \( \mathcal{Y} \) being a compact operator and \( (I - \mathcal{L})^{-1} \in B(L^2_{\text{per}}) \) we have that \( M \) is a compact operator. Then, there is an orthonormal basis \( \{\varphi_k\}_{k=0}^\infty \) of \( L^2_{\text{per}} \) consisting of eigenfunctions of \( M \) with nonzero eigenvalues \( \{\mu_k\}_{k=0}^\infty \) satisfying \( \mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots > 0 \), and \( \mu_k \rightarrow 0 \) as \( k \rightarrow \infty \). Since \( M\varphi_k = \mu_k\varphi_k \in D(L + \mu) \) we have that

\[
\mathcal{L}\varphi_k = \left( \frac{1}{\mu_k} - \mu \right) \varphi_k := \lambda_k \varphi_k.
\]

Thus, there is a sequence of eigenvalues of \( \mathcal{L} \), \( \{\lambda_k\}_{k=0}^\infty \), satisfying \( \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \), and \( \lambda_k \rightarrow \infty \) as \( k \rightarrow \infty \). This argument shows what is desired.

The next step will be centralized in the study of some specific spectral properties of the operator \( \mathcal{L} \). For this, let us define two families of linear operators. They will be related with \( \mathcal{L} \) but with the advantage that both of them are compacts. The results that will be present below are extensions of the results about stability of solitary waves solutions in Albert [2] and Albert and Bona [3] to the periodic case.

For every \( \theta \geq 0 \) define the operator \( S_\theta : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}) \) by considering

\[
S_\theta \alpha(n) = \frac{1}{\omega_\theta(n)} \sum_{j=-\infty}^{\infty} K(n-j) \alpha_j = \frac{1}{\omega_\theta(n)} (K * \alpha)_n,
\]

where \( \omega_\theta(n) = \alpha(n) + \theta + c \), \( K(n) = \varphi_\theta^c(n) \), \( n \in \mathbb{Z} \). Since \( \omega_\theta(n) > 0 \) for all \( n \in \mathbb{Z} \), it follows that the space \( X \) defined by

\[
X = \left\{ \alpha \in \ell^2(\mathbb{Z}); ||\alpha||_{X, \theta} := \left( \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \omega_\theta(n) \right)^{\frac{1}{2}} < \infty \right\}
\]

is a Hilbert space with norm \( ||\alpha||_{X, \theta} \) and corresponding inner product \( \langle \alpha, \beta \rangle_{X, \theta} = \sum_{n=-\infty}^{\infty} \alpha_n \beta_n \omega_\theta(n) \).

**Proposition 3.2.** (a) If \( \alpha \in \ell^2 \) is an eigensequence of \( S_\theta \) for a nonzero eigenvalue, then \( \alpha \in X \).

(b) The restriction of \( S_\theta \) to \( X \) is a compact, self-adjoint operator on \( X \) with respect to the norm \( ||\cdot||_{X, \theta} \).

**Proof.** In fact, we denote the norm in \( X \) simply by \( ||\cdot||_X \), and the operator \( S_\theta \) by \( S \), and \( \mu := \mu_\theta = \frac{1}{\omega_\theta} \). It will be shown that \( S^2g = S(Sg) \in X \), since \( g = \frac{1}{\gamma} Sg = \frac{1}{\gamma} S^2g \);
this will prove the proposition. By the Minkowski, Hölder, and Young inequalities we obtain

\begin{equation}
\|S^2\alpha\|_X = \left\| \sum_j K(\cdot - j)\mu(\cdot)S\alpha(j) \right\|_X = \left\| \sum_j \sum_m K(\cdot - j)\mu(\cdot)K(j - m)\mu(j)m(\cdot) \right\|_X
\end{equation}

\begin{equation}
\leq \sum_j \sum_m \left( \sum_n K(n - j)^2\mu(n) \right)^{\frac{1}{2}} K(j - m)\mu(j)m(\cdot) \leq \|K^2\|_{L^2|\epsilon|} \|\mu\|_{L^2|\epsilon|}^2 \|\alpha\|_{L^2}.
\end{equation}

Since \(|\mu(x)| \leq B(1 + |x|)^{-1}\) for every \(x \in \mathbb{R}\) and some \(B > 0\), all of the quantities in the last expression on the right-hand side in (3.4) are finite. The proof of (a) is complete.

Next, we prove that \(S|_X\) determines a compact and self-adjoint operator. Let

\[ \tilde{K}(n, j) = \frac{K(n - j)}{\omega_\theta(n)\omega_\theta(j)}; \]

then, by the Cauchy–Schwarz inequality we have

\[ \sum_n \sum_j \tilde{K}^2(n, j)\omega_\theta(n)\omega_\theta(j) = \sum_n \sum_j \frac{K^2(n - j)}{\omega_\theta(n)\omega_\theta(j)} = \sum_n (K^2 * \mu)(n)\mu(n) \leq \|K^2\|_{L^2|\epsilon|} \|\mu\|_{L^2|\epsilon|}^2 \leq \|K^2\|_{L^2} \|\mu\|_{L^2}^2 < \infty. \]

That is, \(S_\theta\alpha(n) = \sum_{j=-\infty}^{\infty} \tilde{K}(n, j)\alpha_n\omega_\theta(j)\) is a Hilbert–Schmidt operator, and thus \(S|_X\) is compact. To prove that \(S_\theta\) is self-adjoint it is necessary to observe that \(\omega_\theta\) and \(\tilde{\gamma}_\theta\) being even, the real kernel \(K\) is symmetric.

The next two results are immediate consequences of Proposition 3.2 and the spectral theorem for compact, self-adjoint operators on a Hilbert space.

**Corollary 3.1.** Suppose \(\theta \geq 0\). Then, 1 is an eigenvalue of \(S_\theta\) (as an operator of \(X\)) if and only if \(-\theta\) is an eigenvalue of \(L\) (as an operator of \(L^2\)). Furthermore, both eigenvalues have the same multiplicity.

**Corollary 3.2.** For every \(\theta \geq 0\), \(S_\theta\) has a family of eigensequences \(\{\psi_{i, \theta}\}_{i=0}^{\infty}\) forming an orthonormal basis of \(X\) with respect to the norm \(|\cdot|_{X, \theta}\). The eigensequences correspond to real eigenvalues \(\{\lambda_i(\theta)\}_{i=0}^{\infty}\) whose only possible accumulation point is zero.

In this way, the eigenvalues can be enumerated in order of decreasing absolute value, that is, \(|\lambda_0(\theta)| \geq |\lambda_1(\theta)| \geq |\lambda_2(\theta)| \geq \cdots\).

**4. Positivity properties for periodic travelling-wave solutions.** In this section we give sufficient conditions to obtain properties \((P_1)\) and \((P_2)\) in (1.10).

**Definition 4.1.** We say that a sequence \(\alpha = (\alpha_n)_{n \in \mathbb{Z}} \subseteq \mathbb{R}\) is in the class \(PF(2)\) discrete if

(i) \(\alpha_n > 0\) for all \(n \in \mathbb{Z}\),

(ii) \(\alpha_{n_1 - m_1}\alpha_{n_2 - m_2} - \alpha_{n_1 - m_2}\alpha_{n_2 - m_1} > 0\) for \(n_1 < n_2\) and \(m_1 < m_2\).
The definition above is a particular case of the continuous ones which appear in [2] and [32]; namely, we say that a function \( g : \mathbb{R} \to \mathbb{R} \) is in the class \( PF(2) \) if

(i) \( g(x) > 0 \) for all \( x \in \mathbb{R} \),

(ii) \( g(x_1 - y_1)g(x_2 - y_2) - g(x_1 - y_2)g(x_2 - y_1) > 0 \) for \( x_1 < x_2 \) and \( y_1 < y_2 \).

As an example, consider \( g(x) = \text{sech}^2(x) \).

The next result gives us a sufficient condition for a function \( g \) belonging to the \( PF(2) \) continuous class. This result is very useful for our purpose (see [3]).

**Lemma 4.1.** Suppose \( g \) is a positive, twice-differentiable function on \( \mathbb{R} \) satisfying

\[
\frac{d^2}{dx^2}(\log g(x)) < 0 \quad \text{for} \quad x \neq 0.
\]

Then \( g \in PF(2) \).

The main result of this paper is now presented.

**Theorem 4.1.** Suppose that \( \varphi_c \) is a positive even solution of (1.6) such that \( \varphi_c \geq 0 \) and \( K = \varphi_c^2 \in PF(2) \) discrete. Then \((P_1)\) and \((P_2)\) in (1.10) hold for the operator \( L \) in (1.7).

**Proof.** First, we noticed that, \( S_\theta \) being a compact operator on \( X \), we get a set of eigenvalues \( \{\lambda_i(\theta)\}_{i=0}^\infty \) and the corresponding set of eigenfunctions \( \{\psi_i(\theta)\}_{i=0}^\infty \) which form an orthonormal basis for \( X \). Moreover, we have \( |\lambda_0(\theta)| \geq |\lambda_1(\theta)| \geq |\lambda_2(\theta)| \geq \cdots \).

It will show that the eigenvalues \( \lambda_0(\theta) \) and \( \lambda_1(\theta) \) are positive, distinct, and simple. In fact, since \( S_\theta |_X \) is a compact, self-adjoint operator it follows that

\[
\lambda_0(\theta) = \pm \sup_{||\alpha||_X=1} |\langle S_\theta \alpha, \alpha \rangle_X |.
\]

Let \( \psi(\theta) := \psi \) be an eigensequence of \( S_\theta \) corresponding to \( \lambda_0(\theta) := \lambda_0 \). We will show that \( \psi \) is one-signed; that is, either \( \psi(n) \leq 0 \) or \( \psi(n) \geq 0 \). By contradiction, suppose \( \psi \) takes both negative and positive values. Since by hypotheses the kernel \( K \) is positive we have

\[
S_\theta |\psi|(n) = \frac{1}{\omega_\theta(n)} \sum_{j=-\infty}^{\infty} K(n - j)\psi^+(j) + \frac{1}{\omega_\theta(n)} \sum_{j=-\infty}^{\infty} K(n - j)\psi^-(j)
\]

\[
> \left| \frac{1}{\omega_\theta(n)} \sum_{j=-\infty}^{\infty} K(n - j)\psi^+(j) - \frac{1}{\omega_\theta(n)} \sum_{j=-\infty}^{\infty} K(n - j)\psi^-(j) \right|,
\]

where \( \psi^+ \) and \( \psi^- \) are the positive and negative parts of \( \psi \), respectively. Then,

\[
S_\theta |\psi|(n) > \left| \frac{1}{\omega_\theta(n)} \sum_{j=-\infty}^{\infty} K(n - j)\psi(j) \right| = |S_\theta \psi(n)| = |\lambda_0| |\psi(n)|,
\]

where “>” holds because \( \psi \), by supposition, takes both positive and negative values. From the last inequality we conclude that

\[
\langle S_\theta |\psi|, |\psi| \rangle_{X,\theta} = \sum_{n=-\infty}^{\infty} S_\theta |\psi|(n) |\psi(n)| \omega_\theta(n)
\]

\[
> \sum_{n=-\infty}^{\infty} |\lambda_0| |\psi(n)|^2 \omega_\theta(n) = |\lambda_0| ||\psi||^2_{X,\theta}.
\]
Hence, if we assume that \( \|\psi\|_X = 1 \), we obtain \( \langle S_\theta(|\psi|), |\psi|_X \rangle \lambda_0 \), which contradicts (4.1). Then, there is an eigensequence \( \psi_0 \) which is nonnegative. Since \( K \) is a positive sequence and \( S_\theta(\psi_0) = \lambda_0 \psi_0 \), we have \( \psi_0(n) > 0 \) for all \( n \in \mathbb{Z} \). Now, such a \( \psi_0 \) cannot be orthogonal to any nontrivial one-signed eigensequence in \( X \), and so \( \lambda_0 \) is a simple eigenvalue. Notice that the preceding argument also shows that \( -\lambda_0 \) cannot be an eigenvalue of \( S_\theta \); therefore it follows that \( |\lambda_1| < \lambda_0 \).

Next, we will study the eigenvalue \( \lambda_1(\theta) = \lambda_1 \). But, first, we need some definitions and results. We consider the following set of indices:

\[
\Delta = \{(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}; \ n_1 < n_2\}.
\]

Denoting \( \pi = (n_1, n_2) \) and \( \overline{\pi} = (m_1, m_2) \), we define for \( \pi, \overline{\pi} \in \Delta \) the following sequence:

\[
K_2(\pi, \overline{\pi}) := K(n_1 - m_1)K(n_2 - m_2) - K(n_1 - m_2)K(n_2 - m_1).
\]

By hypothesis \( K \in PF(2) \) discrete it follows that \( K_2 > 0 \). Next, let \( \ell^2(\Delta) \) be defined as

\[
\ell^2(\Delta) = \left\{ \alpha = (\alpha_\pi)_{\pi \in \Delta}; \ \sum_{\Delta} |\alpha_\pi|^2 := \sum_{n_1 \in \mathbb{N}} \sum_{n_2 \in \mathbb{Z}} |\alpha(n_1, n_2)|^2 < +\infty \right\},
\]

and define the operator \( S_{2,\theta} : \ell^2(\Delta) \to \ell^2(\Delta) \) by

\[
S_{2,\theta}g(\pi) = \sum_{\Delta} G_{2,\theta}(\pi, \overline{\pi})g(\overline{\pi}),
\]

where \( G_{2,\theta}(\pi, \overline{\pi}) = \frac{K_2(\pi, \overline{\pi})}{\omega_{\theta}(n_1)\omega_{\theta}(n_2)} \). We also consider the space

\[
W = \left\{ \alpha \in \ell^2(\Delta); \ |\alpha|_{W,\theta} := \left( \sum_{\Delta} |\alpha(\pi)|^2 \omega_{\theta}(n_1)\omega_{\theta}(n_2) \right)^{1/2} < \infty \right\}.
\]

Then \( W \) is a Hilbert space with norm \( |\cdot|_{W,\theta} \) given above and with inner product

\[
\langle \alpha, \beta \rangle_{W,\theta} = \sum_{\Delta} \alpha(\pi)\overline{\beta(\pi)}\omega_{\theta}(n_1)\omega_{\theta}(n_2).
\]

Remark 4.1. (1) We can show, in an analogous way to Proposition 3.2, that \( S_{2,\theta} \vert_W \) is a self-adjoint, compact operator. Therefore, the associated eigenvalues can be enumerated in order of decreasing absolute value, that is, \( |\mu_0(\theta)| \geq |\mu_1(\theta)| \geq |\mu_2(\theta)| \geq \cdots \).

(2) A similar argument can be used to show that \( \mu_0(\theta) := \mu_0 \) is positive and simple and \( |\mu_1| < \mu_0 \).

Definition 4.2. Let \( \alpha^1, \alpha^2 \in \ell^2(\mathbb{Z}) \); we define the wedge product \( \alpha^1 \wedge \alpha^2 \) in \( \Delta \) by

\[
(\alpha^1 \wedge \alpha^2)(n_1, n_2) = \alpha^1(n_1)\alpha^2(n_2) - \alpha^1(n_2)\alpha^2(n_1).
\]

We have the following results from Definition 4.2.

Lemma 4.2. Let \( A = \{\alpha^1 \wedge \alpha^2; \ \text{for} \ \alpha^1, \alpha^2 \in X, \ \alpha^1 \wedge \alpha^2 \in \ell^2(\Delta)\} \). Then \( A \) is dense in \( W \).
Proof. See [31] and [32]. \[ \square \]

**Lemma 4.3.** Let $\alpha^1, \alpha^2 \in l^2(\mathbb{Z})$. Then $S_{2, \theta}(\alpha^1 \land \alpha^2) = S_\theta \alpha^1 \land S_\theta \alpha^2$.

**Proof.** See [31] and [32]. \[ \square \]

In what follows, we will represent by $S_\theta^X$ the restriction of $S_\theta$ on the Hilbert space $X$. We shall use some spectral results in Kato [33]. In fact, we decompose $S_\theta^X$ into the form $S_\theta^X = \lambda_0 P_\theta + Q_\theta$, where $P_\theta$ is the orthogonal projection on $M_0 = [\psi_0], P_\theta Q_\theta = Q_\theta P_\theta = 0$, and $\text{spectr}(Q_\theta) = \text{spectr}(P_\theta) \setminus \{\lambda_0\}$. Moreover, in this case we have that the spectral radius of $Q_\theta$ when restricted to the subspace $N = \ker P_\theta$ is exactly $|\lambda_1|$. Furthermore, we know that the eigenvalue $\lambda_0 = \lambda_0(\theta)$ is differentiable with respect to $\theta \geq 0$. The same argument can be applied to $\mu_0$. Then, as a consequence $\psi_0$ and $\tau_0$ are eigensequences associated with $\mu$ are differentiable eigensequences with respect to $\theta$.

**Lemma 4.4.** (a) In the notation given above we have

$$
\frac{(S_\theta^X) \lambda_0^m}{\lambda_0^m} \rightarrow P_\theta,
$$

where $m \rightarrow +\infty$ on the strong topology of $B(X, X)$.

(b) The statement in part (a) is valid if $S_\theta^X$ is replaced by $S_{2, \theta}$, the eigenvalues $\lambda_0$ and $\lambda_1$ are replaced by $\mu_0$ and $\mu_1$, and $P_\theta, M_0$, and $Q_\theta$ are replaced by appropriate operators and subspaces of $X$.

**Proof.** The proof is analogous as viewed in [2], [3] in the case of sequences. \[ \square \]

**Lemma 4.5.** (a) $\mu_0(\theta) = \lambda_0(\theta) \lambda_1(\theta)$. Then, we can conclude that $\lambda_1 > 0$.

(b) $\lambda_1$ is simple.

**Proof.** (a) In fact, from Lemma 4.3 we have that $\lambda_0 \lambda_1$ is an eigenvalue of $S_{2, \theta}$ whose eigensequence is $\psi_0 \land \psi_1$, where $\psi_1(\theta) := \psi_1$ is the eigensequence associated with $\lambda_1$. Hence $\mu_0 \geq \lambda_0 |\lambda_1|$. Then, since $-\mu_0$ cannot be an eigenvalue of $S_{2, \theta}$, we will show that $\mu_0 \leq \lambda_0 |\lambda_1|$. If $|\lambda_1| < \frac{\mu_0}{\lambda_0}$, let $P_\theta$ be as in Lemma 4.4 and write $W = M_0 \oplus N$. Let $\alpha^1 = r_1 \psi_0 + \omega \gamma^1$ and $\alpha^2 = r_2 \psi_0 + \omega \gamma^2$, where $r_1, r_2 \in \mathbb{R}$ and $\gamma^1, \gamma^2 \in N$. For instance, from the induction principle we have

$$
(S_{2, \theta}) \frac{m}{\mu_0} (\alpha^1 \land \alpha^2)(n_1, n_2) = r_1 \left[ \psi_0(n_1) \left( \frac{S_\theta}{\beta} \right)^m \gamma^2(n_2) - \psi_0(n_2) \left( \frac{S_\theta}{\beta} \right)^m \gamma^2(n_1) \right] \\
+ r_2 \left[ \psi_0(n_2) \left( \frac{S_\theta}{\beta} \right)^m \gamma^1(n_1) - \psi_0(n_1) \left( \frac{S_\theta}{\beta} \right)^m \gamma^1(n_2) \right] \\
+ \left( \frac{S_\theta}{\lambda_0} \right)^m \gamma^1(n_1) \left( \frac{S_\theta}{\lambda_0} \right)^m \gamma^2(n_2) \\
- \left( \frac{S_\theta}{\lambda_0} \right)^m \gamma^1(n_2) \left( \frac{S_\theta}{\lambda_0} \right)^m \gamma^2(n_1),
$$

where $\beta = \frac{\mu_0}{\lambda_0} > |\lambda_1|$. Since $(S_\theta^X)^m \rightarrow P_\theta$ with $P_\theta \equiv 0$ on $N$ and $\beta$ is strictly greater than the spectral radius of the restriction of $S_\theta^X$ to $N$, each term on the right-hand side of the preceding equality tends to zero as $m \rightarrow \infty$. But the set $A$ defined in Lemma 4.2 is dense in $W$. Then, we can conclude from (4.2), after a computation, that $(S_{2, \theta}) \frac{m}{\mu_0} g \rightarrow 0$ strongly. But this contradicts Lemma 4.4. The proof of item (a) is completed.
Thus, \( \psi_0 \) is also one-signed. But this is a contradiction because we have two eigensequences of \( \lambda_1 \) with eigenvalue \( \lambda_0 > 0 \) and \( \lambda_0 < 0 \), respectively. Therefore, either \( \psi_0 \) is odd or \( \psi_0 \) is even, as \( \psi_0 \) is one-signed and \( \psi_1 \) is one-signed. From the definition of wedge product above, that \( 0 \equiv \psi_1 \wedge (\psi_0 \wedge \psi_1) = \lambda_0 \lambda_1 (\psi_0 \wedge \psi_1) \). Since \( \mu_0 > 0 \) and simple we obtain \( \psi_0 \wedge \psi_1 \in |\lambda_0| \). Therefore, either \( \psi_0 \wedge \psi_1 \equiv 0 \) or \( \psi_0 \wedge \psi_1 \equiv \neq 0 \); that is, \( \psi_0 \wedge \psi_1 \) is one-signed. With this fact, if \( \psi_1 \) does not vanish, then it will have at most one zero. Then, \( \psi_1 \) being an even sequence, either \( \psi_1 \equiv 0 \) or else \( \psi_1 \) is one-signed, except possibly in \( n = 0 \). If the second case holds, we have to consider three cases: \( \psi_1(0) = 0 \), \( \psi_1(0) > 0 \), and \( \psi_1(0) < 0 \). If \( \psi_1(0) = 0 \), then we should have, from the fact that \( \psi_1 \) is one-signed and \( \psi(0) \equiv 0 \) for all \( n \in \mathbb{Z} \) because \( \psi_0 \equiv 0 \). Next, if \( \psi_1(0) > 0 \), then for \( n > 0 \) we get \( 0 < \psi_1(n) = \psi_0(n) \psi_1(n) - \psi_0(0) \psi_1(n) < \psi_0(0) \psi_1(n) \) and therefore \( \psi_1(n) > 0 \). The last case is similar to the second one. These considerations make \( \psi_1 \) a one-signed eigensequence of \( S_\theta \) (except possibly in \( n \neq 0 \)), and so the inner product \( \langle \psi_1, \psi_0 \rangle_{X, \theta} \) is also one-signed. But this a contradiction because we have two eigensequences of \( \lambda_1 \) with distinct eigenvalues whose inner product is nonzero. Therefore, \( \psi_1 \equiv 0 \), and so \( \psi_1 \) is odd. A similar argument shows that \( \psi_1 \) can have at most one zero. Of course, this one must be in \( n = 0 \).

The sequence \( \psi_1 \) shown previously was an arbitrary sequence, and this is associated with eigenvalue \( \lambda_1 \). Then, we show that any eigensequence \( \psi \) associated with \( \lambda_1 \) must be odd and \( \psi(n) = 0 \iff n = 0 \). But, two eigensequences of this kind cannot be orthogonal since the product of them is even, and thus \( \lambda_1 \) is simple. This fact completes the proof of the lemma. \( \square \)

We turn back to the proof of Theorem 4.1. Let us consider \( ||\psi_i(\theta)||_{X, \theta} = 1 \) for \( i = 0, 1 \). From Lemma 4.5, \( \mu_0(\theta) = \lambda_0(\theta) \lambda_1(\theta) \) with \( \mu_0 \) and \( \lambda_0 \) differentiable with respect to \( \theta \), and we have that \( \lambda_1 \) is also differentiable with respect to this parameter. Hence, the associated eigensequence \( \psi_1 \) is also differentiable since \( S_\theta \psi_1(\theta) = \lambda_1(\theta) \psi_1(\theta) \). Next, we will show that

\[
(4.3) \quad \frac{d}{d\theta} \lambda_i(\theta) < 0, \quad i = 0, 1, \quad \theta \geq 0.
\]

In fact, writing \( \psi_i(\theta), \ i = 0, 1, \) instead of \( \psi_{i, \theta}(n) \), we have

\[
\frac{d}{d\theta} \lambda_i(\theta) = \frac{d}{d\theta} \sum_{n=-\infty}^{\infty} S_\theta \psi_i(\theta) \psi_i(\theta) \omega(\theta).
\]

Thus,

\[
\frac{d}{d\theta} \lambda_i(\theta) = 2 \sum_{n=-\infty}^{\infty} \frac{d}{d\theta} \psi_i(\theta) S_\theta \psi_i(\theta) \omega(\theta) = 2 \lambda_i(\theta) \sum_{n=-\infty}^{\infty} \left( \frac{d}{d\theta} \psi_i(\theta) \right) \psi_i(\theta) \omega(\theta)
\]

\[
= 2 \lambda_i(\theta) \left\{ \frac{d}{d\theta} \left( \sum_{n=-\infty}^{\infty} \psi_i(\theta)^2 \omega(\theta) \right) - \frac{1}{2} \sum_{n=-\infty}^{\infty} \psi_i(\theta)^2 \right\}
\]

\[
= -\lambda_i(\theta) \sum_{n=-\infty}^{\infty} \psi_i(\theta)^2 < 0.
\]
which shows the affirmation. Next, for $\theta \geq 0$,

$$
\lambda_0(\theta) = r(S_0^X) = ||S_0^X||_{B(X, X)} \leq \left( \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ \frac{K(n-m)}{\omega_0(n)} \right\} \right)^{\frac{1}{2}}
$$

$$
= \left\| K^2 \cdot \frac{1}{\omega_0^2} \right\|_{\ell^1(\mathbb{Z})} \leq \|K\|_{\ell^2(\mathbb{Z})} \left\| \frac{1}{\omega_0} \right\|_{\ell^2(\mathbb{Z})}.
$$

Since $\frac{1}{\omega_0} \to 0$ as $\theta \to +\infty$ and $(\frac{1}{\omega_0})^2 \in \ell^1$ with $|\frac{1}{\omega_0}|^2 \leq (\frac{B}{1+|n|})^2$, for some $B > 0$, $\|\frac{1}{\omega_0}\|_{\ell^2} \to 0$ as $\theta \to +\infty$. Therefore,

$$
\lim_{\theta \to +\infty} \lambda_0(\theta) = 0.
$$

The next step is to show that

$$
\lambda_1(0) = 1.
$$

In fact, since $\frac{d}{dx} \varphi_c$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $\theta = 0$, we can conclude from Corollary 3.1 that $\frac{d}{dx} \varphi_c$ is an eigensequence of $S_0$ with eigenvalue 1. On the other hand, $\frac{d}{dx} \varphi_c(n) = -in\tilde{\varphi}(n)$ is odd and vanishes only at $n = 0$. Since $\psi_1$ is also odd and vanishes at $n = 0$, we must have $\langle \psi_1, \frac{d}{dx} \varphi_c \rangle_{X, \theta} \neq 0$. It follows that $\psi_1$ and $\frac{d}{dx} \varphi_c$ cannot be eigensequences of $S_0$ for distinct eigenvalues. Then, $\psi_1$ and $\frac{d}{dx} \varphi_c$ are associated with the same eigenvalue, and therefore $\lambda_1(0) = 1$. With this fact, since $\lambda_0(0) > \lambda_1(0) = 1$, from (4.3) and (4.4) it follows that there is a unique $\theta_0 \in (0, +\infty)$ such that $\lambda_0(\theta_0) = 1$. Then, from Corollary 3.1, if we consider $\kappa = -\theta_0 < 0$, then $\mathcal{L}$ has a negative eigenvalue which is simple. Now, for $i \geq 2$ and $\theta > 0$ we have from (4.5) that

$$
\lambda_i(\theta) \leq \lambda_1(\theta) < \lambda_1(0) = 1.
$$

It is straightforward to see that 1 cannot be an eigenvalue of $S_0$ for all $\theta \in (0, +\infty) \setminus \{\theta_0\}$, since 1 is an eigenvalue only for $\theta = 0$ and $\theta = \theta_0$. Then we obtain $(P_1)$.

Because $\lambda_1(0) = 1$ and $\lambda_1$ is a simple eigenvalue it follows that $\theta = 0$ is a simple eigenvalue of $\mathcal{L}$ by the Corollary 3.1. This fact shows $(P_2)$ and as a consequence the theorem.

**Remark 4.2.** The Fourier transform in Theorem 4.1 needs to be evaluated in the minimal period of $\varphi_c$ (see section 6).

5. **Stability of periodic travelling-wave solutions.** In this section we are interested in applying the theory in section 4 to obtain the stability of specific periodic travelling waves associated with the KdV, mKdV, and BO equations. Our approach to obtain condition $(P_0)$ in (1.10) will be based on the Poisson summation theorem and the implicit function theorem. This new approach, in the periodic context, will give a simple way to calculate condition $(P_3)$ in (1.10). We start with the definition of stability.

**Definition 5.1.** Let $\varphi$ be a periodic travelling-wave solution with period $2L$ of (1.6), and consider $\tau_r \varphi(x) = \varphi(x + r)$, $x \in \mathbb{R}$, and $r \in \mathbb{R}$. We define the set $\Omega_\varphi \subset H^2_{2L}$, the orbit generated by $\varphi$, as

$$
\Omega_\varphi = \{g; g = \tau_r \varphi \text{ for some } r \in \mathbb{R}\}.
$$
And, for any $\eta > 0$, define the set $U_\eta \subset H^{\frac{m^2}{2L}}$ by

$$U_\eta = \left\{ f : \inf_{g \in U_\eta} \| f - g \|_{H^{\frac{m^2}{2L}}} < \eta \right\}.$$  

With this terminology, we say that $\varphi$ is (orbitally) stable in $H^{\frac{m^2}{2L}}$ by the flow generated by (1.3) if the following hold:

(i) There is $s_0$ such that $H^{s_0} \subseteq H^{\frac{m^2}{2L}}$ and the initial value problem associated with (1.3) is globally well-posed in $H^{s_0}$ (see Theorems 2.2 and 2.3).

(ii) For every $\varepsilon > 0$, there is $\delta > 0$ such that, for all $u_0 \in U_\delta \cap H^{s_0}$, the solution $u$ of (1.3) with $u(0, x) = u_0(x)$ satisfies $u(t) \in U_\varepsilon$ for all $t > 0$.

The proof of the following general stability theorem can be shown by using the techniques in Grillakis, Shatah, and Strauss [27] (see also Angulo [6]).

**Theorem 5.1.** Let $\varphi_c$ be a periodic travelling-wave solution of (1.6), and suppose that part (i) of the definition of stability holds. Suppose also that the operator $L$ defined previously in (1.7) has properties $(P_1)$ and $(P_2)$ in (1.10). Choose $\chi \in L^2_{2L}$ such that $L \chi = \varphi_c$, and define $I = (\chi, \varphi_c)_{L^2_{2L}}$. If $I < 0$, then $\varphi_c$ is stable.

**Remark 5.1.** In our cases the function $\chi$ in Theorem 5.1 satisfies that $\chi = -\frac{d}{dc} \varphi_c$.

Then, we need to verify properties $(P_0)$ and $(P_3)$ in (1.10).

### 5.1. Stability of periodic travelling-wave solutions for the BO equation.

This section is concerned with the stability theory of periodic travelling-wave solutions to the BO equation found initially by Benjamin in [13]. Next, we will present an interesting method for obtaining an explicit solution to the BO equation in (3.1), by using the Poisson summation theorem. In fact, consider the following equation:

$$\mathcal{H} \phi'_\omega + \omega \phi_\omega - \frac{1}{2} \phi^2_\omega = 0.$$  

This equation determines solitary travelling-wave solutions to the BO equation on $\mathbb{R}$ in the form

$$\phi_\omega(x) = \frac{4\omega}{1 + \omega^2 x^2}, \quad \omega > 0.$$  

Its Fourier transform is given by

$$\hat{\phi}_\omega^R(x) = 4\pi e^{-2\pi |x|}.$$  

Then, by the Poisson summation theorem, we obtain that

$$\psi_\omega(x) = \sum_{n=-\infty}^{+\infty} \phi_n(x + 2Ln) = \frac{2\pi}{L} \sum_{n=-\infty}^{+\infty} \varepsilon_n e^{-\pi |n|} e^{\pi i n x}$$

$$= \frac{2\pi}{L} \sum_{n=0}^{+\infty} \varepsilon_n e^{-\pi n} \cos \left( \frac{n\pi x}{L} \right) = \frac{2\pi}{L} \text{Re} \left[ \coth \left( \frac{\pi}{2\omega L} + \frac{i\pi x}{2L} \right) \right]$$

$$= \frac{2\pi}{L} \left( \frac{\text{senh} \left( \frac{\pi}{2\omega L} \right)}{\cosh \left( \frac{\pi}{2\omega L} \right) - \cos \left( \frac{\pi x}{2L} \right)} \right),$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n = 1, 2, 3, \ldots \end{cases}$$
Let \( \varphi_c, c > 0 \), be a smooth periodic solution of the first equation in (3.1). Thus, \( \varphi_c \) can be expressed as a Fourier series

\[
\varphi_c(x) = \sum_{n=-\infty}^{+\infty} a_n e^{\frac{i n \pi x}{L}}.
\]

(5.2)

Substituting the expression above into the BO equation in (3.1), we get

\[
\left[ \frac{\pi |n|}{L} + c \right] a_n = \frac{1}{2} \sum_{m=-\infty}^{+\infty} a_{n-m} a_m.
\]

Next, from (5.1) we consider \( a_n = \frac{2\pi}{L} e^{-\gamma|n|}, \ n \in \mathbb{Z}, \gamma \in \mathbb{R} \). Substituting \( a_n \) into the last identity we have

\[
\sum_{m=-\infty}^{+\infty} a_{n-m} a_m = \frac{4\pi^2}{L^2} e^{-\gamma|n|} \left[ |n| + 1 + 2 \sum_{k=1}^{+\infty} e^{-2\gamma k} \right] = \frac{4\pi^2}{L^2} e^{-\gamma|n|} (|n| + \coth \gamma).
\]

Then, we conclude that

\[
c + \frac{\pi |n|}{L} = \frac{2\pi}{L} \cdot \frac{1}{2} (|n| + \coth \gamma).
\]

(5.3)

We denote \( \gamma = \frac{\pi}{\omega L} \) and consider \( c > \frac{\pi}{L} \). Then, if we choose \( \omega = \omega(c) > 0 \) such that \( \tanh(\gamma) = \frac{c}{\pi L} \), we obtain from (5.3) that \( \psi_{\omega(c)} = \varphi_c \) (hence, \( \varphi_c \) is given by (5.1)). Therefore, we obtain that \( \varphi_c \) has the form in (1.13) with \( \gamma > 0 \) satisfying \( \tanh(\gamma) = \frac{c}{\pi L} \).

Thus, from (5.1) we have that \( \varphi_c > 0 \), and \( \gamma := \gamma(c) = \tanh^{-1} \left( \frac{c}{\pi L} \right) \) being a differentiable function for \( c > \frac{\pi}{L} \), it follows that

\[
c \in \left( \frac{\pi}{L}, +\infty \right) \mapsto \varphi_c \in H^2_{2L}
\]

is a smooth curve of periodic travelling-wave solutions for the BO equation for all \( n \in \mathbb{N} \). Then, by defining \( \chi \) in Theorem 5.1 as \( \chi = -\frac{d}{dc} \varphi_c \), we obtain from the first equation in (3.1) that \( L \chi = \varphi_c \). Then \( I = \langle \chi, \varphi_c \rangle_{L_{2L}^2} \) becomes

\[
I = -\frac{1}{2} \frac{d}{dc} \| \varphi_c \|_{L_{2L}^2}^2.
\]

(5.4)

We will show that \( I < 0 \). Indeed, from (5.2) and (5.1) we get

\[
\varphi_c(x) = \frac{2\pi}{L} \sum_{n=-\infty}^{+\infty} e^{-\gamma|n|} e^{\frac{i n \pi x}{L}};
\]

then, from the Parseval theorem we conclude that

\[
I = \frac{1}{2} \frac{d}{dc} \| \varphi_c \|_{L_{2L}^2}^2 = \frac{1}{2} \frac{d}{dc} \| \varphi_c \|_{L^2}^2 \cdot 2L = -\frac{1}{2} \frac{d}{dc} \left( \frac{4\pi^2}{L^2} \sum_{n=-\infty}^{+\infty} e^{-2\gamma|n|} \right) \cdot 2L
\]

\[
= -\frac{4\pi^3}{c^2 L^3} \left( \frac{1}{1 - \left( \frac{c}{\pi L} \right)^2} \right) \left( \sum_{n=-\infty}^{+\infty} |n| e^{-2\gamma|n|} \right) \cdot 2L < 0.
\]

(5.6)
Finally, we will verify that conditions \((P_1)\) and \((P_2)\) are true for the operator

\[ L_{BO} = \mathcal{H}\partial_x + c - \varphi_c. \]

Let \( \widehat{\varphi}_c(n) = \frac{2\pi}{L} e^{-\gamma|n|} \) be the Fourier coefficients of \( \varphi_c \). In Albert [2] it has already been seen that the function \( f(x) = e^{-\gamma|x|} \) belongs to the \( PF(2) \) class in the continuous case, and so \( \widehat{\varphi}_c \) is in the \( PF(2) \) class in the sense of Definition 4.1. Hence, we obtain from Theorem 2.3 the stability of the periodic solutions (1.13) in \( H^2_L \) by the periodic flow of the BO equation.

### 5.1.1. Stability of the constant solutions

To complete the investigation about periodic travelling-wave solutions for the BO equation, we will study the stability of the constant solutions. Hence, if \( \varphi_c(\xi) \equiv \tau \) is a constant solution to the BO equation, we have \( \varphi_c \equiv 2c \) and \( \varphi_c \equiv 0 \) as solutions. We consider only the first case. The next result will resume our purpose.

**Proposition 5.1.** Let \( L > 0 \) and \( c > 0 \) be given. Consider \( \psi_0 \equiv 2c \) to be a nontrivial constant solution of (3.1). Then, \( \psi_0 \) is stable in \( H^2_L([-L,L]) \), provided \( c < \frac{\pi}{L} \).

**Proof.** The proof of this proposition follows from standard ideas (see [10], [15]) and from the following nonlocal Poincaré–Wirtinger-type inequality: for \( f \in H^2_L \) such that \( \int_{-L}^L f(x)dx = 0 \), we have \( \int_{-L}^L [D^2 f(x)]^2 dx \geq \frac{\pi^2}{L^2} \int_{-L}^L f^2(x) dx \), where \( D = \mathcal{H}\partial_x \) (see the appendix for a proof of this inequality).

### 5.2. Stability of periodic travelling-wave solutions for the mKdV equation

Next, we will establish the existence of a smooth curve of periodic travelling-wave solutions for the mKdV equation

\[ u_t + 3u^2u_x + u_{xxx} = 0, \]

of the form \( u(x,t) = \varphi(x - ct) := \varphi_c(\xi) \), where \( \xi = x - ct \), \( c \in \mathbb{R} \), and is of period \( L \). The equation which determines the periodic travelling-wave solutions is

\[ \varphi_c'' + \varphi_c^3 - c\varphi_c = 0. \]

(5.8)

Next, we obtain an explicit solution for (5.8) using the Poisson summation theorem. It considers for \( \omega > 0 \) the solitary wave solutions for the mKdV equation on \( \mathbb{R} \):

\[ \phi_\omega(x) = \sqrt{2\omega} \text{sech}(\sqrt{\omega}x), \quad x \in \mathbb{R}. \]

Its Fourier transform is \( \widehat{\phi}_\omega(x) = \sqrt{2}\pi \text{sech}(\frac{\pi x}{2\sqrt{\omega}}) \), where \( \omega > 0 \), and it will be chosen later. From the Poisson summation theorem we obtain the following periodic function of period \( L \):

\[ \psi_\omega(\xi) = \frac{\sqrt{2}\pi}{L} \sum_{n=0}^\infty \varepsilon_n \text{sech} \left( \frac{\pi n}{2\sqrt{\omega}L} \right) \cos \left( \frac{2\pi n\xi}{L} \right), \]

(5.9)

where

\[ \varepsilon_n = \begin{cases} 
1, & n = 0, \\
2, & n = 1, 2, 3, \ldots
\end{cases} \]
On the other hand, it considers the Fourier expansion of the Jacobi elliptic function *dnoidal*, $dn$, of period $L$ (see [19], [35], [42]),

$$
\frac{2K}{L} \text{dn} \left( \frac{2K \xi}{L}; k \right) = \frac{\pi}{L} + \frac{4\pi}{L} \sum_{n=1}^{+\infty} \frac{q^n}{1 + q^{2n}} \cos \left( \frac{2n\pi \xi}{L} \right),
$$

where $K = K(k)$ is the complete elliptic integral of the first kind and $q = e^{(-\pi K')}$, which is called the “nome.” Here, $K'(k) = K(\sqrt{1-k^2})$. We can conclude that

$$
\frac{q^n}{1 + q^{2n}} = \frac{1}{2} \text{sech} \left( \frac{n\pi K'}{K} \right).
$$

Therefore,

$$
\frac{2K}{L} \text{dn} \left( \frac{2K \xi}{L}; k \right) = \frac{\pi}{L} + \frac{2\pi}{L} \sum_{n=1}^{+\infty} \text{sech} \left( \frac{n\pi K'}{K} \right) \cos \left( \frac{2n\pi \xi}{L} \right).
$$

Because of the shape of the series that determines $\phi_\omega$ given above (see [35]), let $\phi_c(\xi) = \eta \text{dn}(\frac{\eta \xi}{\sqrt{2}}; k)$ be a periodic solution of period $L$ for (5.8), with $\eta > 0$ and $k \in (0, 1)$ fixed. Then, the following identities should be satisfied:

\begin{equation}
(5.10) \quad c = \frac{\eta^2}{2} (1 + k'^2) \quad \text{and} \quad \eta = \frac{2\sqrt{2}K(k)}{L}
\end{equation}

with $k'^2 = 1 - k^2$. Thus, for $k \in (0, 1)$ we should have that $\eta \in (\sqrt{c}, \sqrt{2c})$ and from the asymptotic properties of $K$ that $c > \frac{2k^2}{L^2}$. Next, for $k \in (0, 1)$ fixed, $\eta$ is immediately defined from (5.10), and so

\begin{equation}
(5.11) \quad c = \frac{4K^2(k)}{L^2} (2 - k^2) > \frac{2\pi^2}{L^2}
\end{equation}

since $k \to K^2(k)(2 - k^2)$ is a strictly increasing function. Therefore, with $k \in (0, 1)$ and $c$ defined in (5.11) we define $\omega = \omega(c)$ as

$$
\omega = \frac{c}{16(2 - k^2)K'^2(k)}.
$$

Therefore, from (5.9) it follows that $\psi_{\omega(c)} = \varphi_c$ is a solution of (5.8).

Next, it is necessary to build a smooth curve, $c \to \varphi_c$, of dnoidal wave solutions. Initially, we obtain the following a priori estimate for the fundamental period of $\varphi_c$. Namely,

\begin{equation}
(5.12) \quad T_{\varphi_c}(\eta) = \frac{2K(k(\eta))}{\sqrt{c}} \sqrt{2 - \frac{2}{\sqrt{c}}} > \frac{\sqrt{2\pi}}{\sqrt{c}},
\end{equation}

where $k^2(\eta) = 2 - \frac{2}{\sqrt{c}}$. In fact, for $\eta \to \sqrt{c}$ it follows that $k \to 0^+$ and thus $T_{\varphi_c}(\eta) \to \frac{\pi\sqrt{2}}{\sqrt{c}}$. For $\eta \to \sqrt{2c}$ we have $k \to 1^-$ and therefore $T_{\varphi_c}(\eta) \to +\infty$. But, $\eta \mapsto T_{\varphi_c}(\eta)$ being a strictly increasing function (see Theorem 2.1 in Angulo [9]) we get (5.12).

**Remark 5.2.** The function $\varphi_c$ obtained previously, that is,

\begin{equation}
(5.13) \quad \varphi_c(\xi) = \eta \text{dn} \left( \frac{\eta}{\sqrt{2}} \xi; k \right),
\end{equation}

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is a positive function and has been built by the periodization of the solitary wave solution associated with (5.7). Thus, it is natural to ask if we can again obtain this solitary wave. Indeed, this fact can be determined by (5.13), since for \( \eta \to \sqrt{2c} \) we have \( k \to 1^- \) and then \( \text{dn}(u; 1^-) = \text{sech}(u) \). Hence we have formally that \( \varphi_c(\xi) = \sqrt{2c} \text{ sech}(\sqrt{2c} \xi) \). The other limit case, that is, \( \eta \to \sqrt{c} \), we have \( k \to 0^+ \) and so \( \text{dn}(u; 0^+) = 1 \); then we get \( \varphi_c(\xi) = \sqrt{c} \), the nontrivial constant solutions for the mKdV.

Now, we construct a family of dnoidal waves solutions with period \( L \). Let \( c > 0 \) such that \( \sqrt{c} > \frac{\pi}{2L} \). Since \( \eta \in (\sqrt{c}, \sqrt{2c}) \to T_{\varphi_c}(\eta) \) is a strictly increasing mapping, it follows from (5.12) and from Theorem 2.1 in Angulo [9] that there is a unique \( \eta \equiv \eta(c) \in (\sqrt{c}, \sqrt{2c}) \) such that the fundamental period of the dnoidal wave \( \varphi_c \) will be \( T_{\varphi_c}(\eta(c)) = L \). Moreover, we have \( c \in (2\pi^2/L^2, +\infty) \to \varphi_c \in H_{per}^2([0, L]) \) is a smooth curve, \( c \in (2\pi^2/L^2, +\infty) \to \eta = \eta(c) \) is a strictly increasing function, and its derivative with respect to the velocity \( c \) is given by

\[
\frac{dc}{d\eta} = \frac{\eta}{2c} + \frac{k^2 k'^2 \eta^3 (2 - k^2) K}{\sqrt{c^7 (2 - k^2) E - 2(1 - k^2) K}}.
\]

The next result gives us a relation between the velocity of the solitary and periodic waves associated with the mKdV, and it will be useful later.

**Theorem 5.2.** We consider the mapping \( \omega : (2\pi^2/L^2, +\infty) \to \mathbb{R} \), given by

\[
\omega(c) = c/(16(2 - k^2)K'^2(k));
\]

then \( \frac{d\omega}{dc} > 0 \).

**Proof.** Indeed, \( \frac{d\omega}{dc} = [4(2 - k^2)K'^2 + 8K'c \frac{dc}{dk}(kK' - (2 - k^2) \frac{dk'}{dc})]/[16(2 - k^2)K'^4] \).

Since \( \frac{dk'}{dc} = -(\frac{E' - k^2 K'}{kk'^2}) < 0 \), it suffices to show that \( \frac{dk}{dc} > 0 \). So, since

\[
\frac{dk}{dc} = \frac{2}{\eta^3} \left( 2c \frac{d\eta}{dc} - \eta \right),
\]

we need to verify the sign of the expression \( 2c \frac{d\eta}{dc} - \eta \). Next, we calculate the exact value of \( \frac{d\eta}{dc} \). In fact,

\[
\frac{d\eta}{dc} = \frac{\eta}{2c} + \frac{k^2 k'^2 \eta^3 (2 - k^2) K}{\sqrt{c^7 (2 - k^2) E - 2(1 - k^2) K}}.
\]

Thus, \( 2c \frac{d\eta}{dc} - \eta = 2cA \) and therefore \( \frac{dk}{dc} > 0 \). This fact completes the proof of the theorem. \( \square \)

Next, we will show that \( \bar{\varphi}_c > 0 \) and \( K = \bar{\varphi}_c^2 \) belongs to \( PF(2) \) discrete. It is easy to see that \( \bar{\varphi}_c > 0 \) because of the form of the Fourier coefficients of \( \varphi_c \) given by (5.9). Moreover, \( \bar{\varphi}_c \in PF(2) \) discrete because the function \( f(x) = \mu \text{sech}(\nu x) \) belongs to \( PF(2) \) continuous (see [2]). So, since the convolution of even sequences in \( PF(2) \) discrete is a sequence in \( PF(2) \) discrete (see [32]) we can conclude that \( K = \bar{\varphi}_c^2 \in PF(2) \) discrete. Now, by choosing \( \chi = -\frac{d}{dc}\varphi_c \) we have that \( L\chi = \varphi_c \). Then, by the Parseval theorem, \( I = -\frac{1}{2} \frac{d}{dc} \|\varphi_c\|^2_{L^2_{per}} = -\frac{1}{2} \frac{d}{dc} \|\varphi_c\|^2_{L^2} \). Since \( \|\varphi_c\|^2_{L^2} =
\[ 2 \frac{\pi^2}{L^2} \sum_{n=-\infty}^{+\infty} \text{sech}^2 \left( \frac{\pi n}{\sqrt{\omega(c)L}} \right) \], we have that

\[
\frac{d}{dc} ||\hat{\varphi}_c||^2_{L^2} = \frac{C_1(L)}{\sqrt{\omega(c)L}} \frac{d\omega}{dc} \sum_{n=-\infty}^{+\infty} \text{sech}^2 \left( \frac{\pi n}{\sqrt{\omega(c)L}} \right) \text{ntgh} \left( \frac{\pi n}{\sqrt{\omega(c)L}} \right). 
\]

Since \((n \text{ tgh}(\frac{\pi n}{\sqrt{\omega(c)L}}))_{n \in \mathbb{Z}}\) is a positive sequence we have from Theorem 5.2 that \(\frac{d}{dc} ||\hat{\varphi}_c||^2_{L^2} > 0\). Thus, we obtain that the dnoidal wave \(\varphi_c\) in (5.14) is stable in \(H^1_{per}([0, L])\) by the flow of the mKdV.

5.3. Stability of periodic travelling-wave solutions for the KdV equation. Now, we apply the results obtained previously to the proof of the stability of periodic travelling-wave solutions of cnoidal type associated with the KdV equation and satisfying

\[ \varphi''_c + \frac{1}{2} \varphi^2_c - c \varphi_c = 0. \] (5.15)

We consider the solitary wave solutions \(\phi_\omega(x) = 3\omega \text{sech}^2 \left( \frac{\sqrt{\omega} x}{L} \right)\), whose Fourier transform is given by \(\hat{\phi}_\omega(x) = \frac{12\pi}{L} \frac{\text{senh} \left( \frac{\pi x}{\sqrt{\omega L}} \right)}{\sqrt{\omega L}}\); then from the Poisson summation theorem we consider

\[ \psi_\omega(\xi) = \frac{12\sqrt{\omega}}{L} + \frac{12\pi}{L^2} \sum_{n \neq 0} ncsh \left( \frac{\pi n}{\sqrt{\omega L}} \right) e^{\frac{2\pi i n \xi}{L}}. \] (5.16)

Since \(\omega\) is arbitrary, consider \(\omega := \omega(k)\) such that \(\sqrt{\omega(k)} = \frac{K(k)}{K(k')}L\), \(k \in (0, 1)\), \(k'^2 = 1 - k^2\). Then, we obtain

\[ \psi_{\omega(k)}(\xi) = \frac{12\sqrt{\omega(k)}}{L} + \frac{24\pi}{L^2} \sum_{n=1}^{+\infty} nsch \left( \frac{\pi n K'}{K} \right) \cos \left( \frac{2\pi n \xi}{L} \right). \] (5.17)

Now, we invoke the Fourier expansion of \(\text{dn}^2\) (see [19], [42]), that is,

\[ K^2 \left( \text{dn}^2 \left( \frac{2K\xi}{L}; k \right) - \frac{E}{K} \right) = 2\pi \sum_{n=1}^{+\infty} \frac{nq^n}{1 - q^{2n}} \cos \left( \frac{2\pi n \xi}{L} \right), \]

where \(q = e^{-\left( \frac{\pi K'}{K} \right)}\). We can conclude that

\[ \frac{q^n}{1 - q^{2n}} = \frac{1}{2} \text{csch} \left( \frac{n\pi K'}{K} \right). \]

Then, we get from (5.17)

\[ \psi_{\omega(k)}(\xi) = \frac{12\sqrt{\omega(k)}}{L} + \frac{24K^2}{L^2} \left( \text{dn}^2 \left( \frac{2K\xi}{L}; k \right) - \frac{E}{K} \right) \] (5.18)

for \(k \in (0, 1)\).
Next, because of the equality (5.18), we consider \( \varphi_c(\xi) = a + b \left( \text{dn}^2(dx; k) - \frac{E}{K} \right) \) a periodic travelling-wave solution for (5.15) of period \( L \). Then, the following nonlinear system is obtained:

\[
\begin{align*}
\frac{b^2}{2} - 6d^2b &= 0, \\
\frac{a^2}{2} - \frac{abE}{K} + \frac{b^2}{2} \left( \frac{E}{K} \right)^2 - ac - \frac{b^2E}{K} - 2bd^2k^2 &= 0.
\end{align*}
\]

(5.19)

Since \( \varphi_c \) is periodic of period \( L \) it follows that \( d = \frac{2K(k)}{L} \). Then, from the first equation of the system above we have that \( b = \frac{48K^2k^2}{L^2} \). Substituting those values at the second equation we get

\[
\frac{a^2}{2} - \frac{abE}{K} + \frac{b^2}{2} \left( \frac{E}{K} \right)^2 - ac - \frac{b^2E}{K} - 2bd^2k^2 = 0.
\]

(5.20)

From the third equation in (5.19) and the value of \( c \) in (5.20) we have the quadratic equation in terms of \( a \),

\[
a^2 + \frac{32K^2}{L^2} \left[ (1 + k^2)K - 3E \right] a - \frac{(1 + k^2)1536K^3E}{L^4} + \frac{768K^4k^2}{L^4} + \frac{2304K^2E^2}{L^4} = 0,
\]

(5.21)

whose positive solution is

\[
a = -\frac{16K}{L^2} \left[ (1 + k^2)K - 3E \right] + \frac{16K^2}{L^2} \sqrt{1 - k^2 + k^4}.
\]

Thus, the value of \( c \) is \( c = \frac{16K^2}{L^2} \sqrt{1 - k^2 + k^4} \). Hence, for \( k \in (0, 1) \) we have that \( c \in \left( \frac{4K}{L}, +\infty \right) \). Therefore, writing \( \varphi_c \) in a convenient form, in terms of \( \text{cn}^2 \), we obtain

\[
\varphi_c(\xi) = \frac{16K^2}{L^2} \left[ \sqrt{1 - k^2 + k^4 + 1 - 2k^2} + \frac{48K^2k^2}{L^2} \text{cn}^2 \left( \frac{2K}{L}; k \right) \right].
\]

We can see that this formula is the same that was obtained by Angulo in [7] and it can be rewritten as

\[
\varphi_c(\xi) = \beta_2 + (\beta_3 - \beta_2)\text{cn}^2 \left( \sqrt{\frac{\beta_3 - \beta_1}{12}}\xi; k \right),
\]

where

\[
\beta_2 = \frac{16K^2}{L^2} \left[ \sqrt{1 - k^2 + k^4 + 1 - 2k^2} \right], \quad \beta_3 = \frac{16K^2}{L^2} \left[ \sqrt{1 - k^2 + k^4 + 1 + k^2} \right],
\]

and \( \beta_1 \) is such that

\[
\beta_3 - \beta_1 = \frac{48K^2}{L^2}.
\]

By making a similar analysis such as in the case of the mKdV equation, we can obtain a smooth curve of positive cnoidal waves with the form in (5.22), \( c \in \left( \frac{4K}{L}, +\infty \right) \).
there is a unique $c \in \left(\frac{4\pi^2}{L^2}, +\infty\right)$ such that $k := k(c)$ is a strictly increasing smooth function (see [7]) for all $n \in \mathbb{N}$. Moreover, we can determine that for $k \in (0, 1)$ there is a unique $c \in \left(\frac{4\pi^2}{L^2}, +\infty\right)$ such that $k(c) = k$. Therefore, the function $\omega(k)$ defined above can be expressed as a function of $c$, $\omega = \omega(k(c))$, and it is a strictly increasing function (it will be seen later). Then, since $K(k) \in (0, +\infty)$ it follows that for $c \in \left(\frac{4\pi^2}{L^2}, +\infty\right)$ we obtain $\omega(k(c)) \in (0, +\infty)$. Therefore, the mapping $c \in \left(\frac{4\pi^2}{L^2}, +\infty\right) \mapsto \omega(k(c)) \in H^2_{per}(\mathbb{R})$ is a smooth curve for all $n \in \mathbb{N}$. Next, we note that $2\psi_{\omega(k(c))}(\xi) - \varphi_c(\xi) = \frac{24\sqrt{\omega(k(c))}}{L} - a(k(c))$, where for $k = k(c)$

$$a(k) = \frac{16K^2}{L^2} \left[ \sqrt{1 - k^2 + k^4 + 2 - k^2 + 3\frac{E}{K}} \right].$$

Thus, for $s(k(c)) = a(k(c)) - \frac{24\sqrt{\omega(k(c))}}{L}$, we can write $\varphi_c(\xi) = s(k(c)) + \psi_{\omega(k(c))}(\xi)$. Therefore, we obtain immediately that the Fourier coefficients of $\varphi_c$ are

$$\hat{\varphi}_c(n) = \begin{cases} a(k), & n = 0, \\ \frac{12\pi}{L^2} \text{ncsch} \left( \frac{\pi n}{\sqrt{\omega(k)L}} \right), & n \neq 0. \end{cases}$$

**Remark 5.3.** After some calculations, we can obtain that $s(k)$ is a positive function defined in $(0, 1)$. Making use of Maple, we can also determine that $s(k)$ does not have any root on the extremes of the interval $(0, 1)$. We can also determine that the function $a(k)$ is a positive strictly increasing function (see Figure 1).

Since $s(k) > 0$ and the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{12\pi}{L^2} \text{ncsch}(\frac{\pi x}{\sqrt{\omega(k)L}})$, belongs to $PF(2)$ in the continuous case, we can use Lemma 4.1 for obtaining that $\varphi_c$ belongs to the $PF(2)$ discrete case. Indeed, since

$$a(k) > \frac{24\sqrt{\omega(k)}}{L} > \frac{12\sqrt{\omega(k)}}{L} = f(0) > f(x), \quad x \neq 0,$$

we can redefine $f$ by a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = a(k)$, $h(x) \equiv f(x)$ on $(-\infty, -1] \cup [1, +\infty)$ and on the interval $(-1, 1)$ we “complete” $f$ in a differentiable
way, such that $h$ belongs to $PF(2)$ continuous. Therefore, the sequence to be obtained (if we look only at the set of integers numbers) will be $h(n) = \hat{\varphi}_c(n)$. 

Next, let $\chi = -\frac{d}{dc}\varphi_c$ such that $L\chi = \varphi_c$. Then by the Parseval theorem, it follows that $I = -\frac{L}{2} \frac{d}{dc} \left( \|\hat{\varphi}_c\|_2^2 \right)$. Hence,

$$
\frac{d}{dc} \|\hat{\varphi}_c\|_2^2 \bigg|_{L_{per}} = C_1 a(k) \frac{da}{dk} \frac{dk}{dc} + C_2 \frac{d\omega}{dk} \frac{dk}{dc} \sum_{n=\infty}^{+\infty} \frac{n^3 \text{csch}^2 \left( \frac{\pi n}{L \sqrt{\omega(k)}} \right) \coth \left( \frac{\pi n}{L \sqrt{\omega(k)}} \right)}{b_n},
$$

where $C_1 := C_1(L)$, $C_2 := C_2(L) > 0$. Next, we need to show only that the quantities $\frac{da}{dk}$ and $\frac{d\omega}{dk}$ are positive because $k := k(c)$ is a strictly increasing function and $(b_n)_{n \in \mathbb{Z}}$ is obviously a positive sequence. Hence, we have

$$
\frac{d\omega}{dk} = 2 \frac{K \left( \frac{dK}{dk} K' - K \frac{dK'}{dk} \right)}{K'^3}.
$$

Since $\frac{dK}{dk} > 0$ and $\frac{dK'}{dk} < 0$ we get that $\frac{dk}{dc} > 0$. By making use of a similar argument, we can also show that $\frac{da}{dk} > 0$ because we have that

$$
a(k) = \frac{16 K(k)^2}{L^2} \left[ \sqrt{1 - k^2} + k^2 - k^4 \right] + 48 \frac{E(k)K(k)}{L^2}.
$$

Therefore, $I < 0$ and the positive cnoidal waves $\varphi_c$ are stable in $H^1_{per}([0, L])$ by the periodic flow of the KdV equation.

6. Comments. In this section we make some basic remarks about the results contained in the body of this paper.

6.1. General perturbations. In contrast to the case of solitary waves for which the natural class of disturbance in the stability problem is that of localized disturbances, for periodic waves there are several classes of disturbance for which stability needs to be addressed. Here we consider periodic perturbations with the same fundamental period of the periodic travelling waves. For disturbance, for example, with a double period, stability results remain open in the context of KdV-type equations. If we consider the KdV equation, our stability result in $H^1_{per}([0, L])$ was based on the spectral structure of the operator $L_{CN} = -\frac{d^2}{dx^2} + c - \varphi_c$, with $\varphi_c$ defined in (5.22). Now, if we consider this operator with domain $H^2_{per}([0, 2L])$, then the number of negative eigenvalues will be exactly 3 (see [10]). Moreover, since the function $\frac{1}{L} \int_{-L}^{L} \varphi_c^2(x)dx$ is even positive, the abstract setting of Grillakis, Shatah, and Strauss in [27] and [28] cannot be applied. We note that in the case of the focusing Schrödinger equation

$$
iu_t + u_{xx} + |u|^2u = 0,$$

Angulo in [9] showed an instability result for cnoidal waves solutions when the class of periodic disturbance is two times the minimal period of the periodic travelling wave in question (see also Gallay and Hărăguş [24], [25] for recent new results of stability for periodic travelling waves of Schrödinger equations).
6.2. Property $PF(2)$ discrete. We consider the BO equation (but we can do
an analogous analysis with the other two equations): then we have seen throughout
this paper that the spectral properties of the operator $L = \mathcal{H}_c - \varphi_c$ were obtained
from the fact that the Fourier coefficients of the periodic travelling-wave solution given
by (1.13) are in $PF(2)$. The Poisson summation theorem was used to find such a wave
with a minimal period $2L$. Moreover, the Fourier coefficients were calculated with
this period. If we double the period, that is, if we consider $L$ with domain $D(L) = H^1_{2L}$, the property $PF(2)$ is not satisfied to $\tilde{K} = \hat{\varphi}_c^{(4L)}$, where $\hat{\varphi}_c^{(4L)}$ denotes the
periodic Fourier transform of $\varphi_c$ but with period $4L$. Indeed, we consider the Fourier
expansion in the form $\varphi_c(x) = \sum_{n=0}^{+\infty} \hat{\varphi}_c^{(4L)}(n) \cos \left(\frac{n\pi x}{2L}\right)$. Since $\varphi_c(0) = \varphi_c(2L)$,
$\sum_{n=0}^{+\infty} \hat{\varphi}_c^{(4L)}(n)(\cos(n\pi) - 1) = 0$. Thus, $-2\sum_{n=2k+1}^{+\infty} \hat{\varphi}_c^{(4L)}(n) = 0$. Therefore,
$\tilde{K} = \hat{\varphi}_c^{(4L)}$ cannot belong to $PF(2)$. Then, we cannot affirm anything about the
stability of the wave $\varphi_c$ when this case is considered. So, Theorem 4.1 cannot be
applied here.

In Theorem 4.1, the Fourier transform needs to be evaluated in the minimal period
for the solution $\varphi_c$. In fact, let $L$ be this minimal period, and we evaluate the Fourier
transform of $\varphi_c$ as being of period $2L$; then
\[
\hat{\varphi}_c(k) = \frac{1}{2L} \int_{-L}^{L} \varphi_c(x)e^{-\frac{ixk}{L}} dx = \frac{1}{2L} \int_{-L}^{L} \varphi_c(x+L)e^{-\frac{ixk}{L}} dx \]
\[
= \frac{1}{2L} \int_{0}^{2L} \varphi_c(y)e^{-\frac{i\pi y}{L}} e^{i\pi k} dy = (-1)^k \hat{\varphi}_c(k).
\]
Then, for $k$ being odd we have $\hat{\varphi}_c(k) = 0$. Thus, we cannot apply our theory if the
minimal period is not fixed.

6.3. Positivity of the periodic travelling waves. Another fact that deserves
a special mention is about the condition that the solution $\varphi_c$ in Theorem 4.1 needs
to be a positive solution. Indeed, the classical Fourier theorem (see [30]) told us that
this is necessary. Suppose without loss of generality that $\varphi_c(0) = 0$. Then, $\varphi_c$ being
smooth it follows that $\varphi_c(0) = \sum_{n=-\infty}^{+\infty} \hat{\varphi}_c(n) = 0$. In other words some Fourier
coefficient of $\varphi_c$ must be negative.

6.4. Stability and instability of periodic travelling waves for the critical
KdV and NLS. In a forthcoming paper [11] we apply the theory in section 4 to
obtain the stability/instability of a special family of periodic travelling-wave solutions
for the critical KdV equation
\[
u_t + 5u^4u_x + u_{xxx} = 0
\]
and for the critical nonlinear Schrödinger (NLS) equation
\[
iu_t + u_{xx} + |u|^4u = 0.
\]

7. Appendix.

7.1. Jacobi elliptic functions. We establish some basic properties of Jacobian
elliptic integrals (see [18] and [19]). The normal elliptic integral of the first kind is
\[
\int_0^\varphi \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = F(\varphi, k),
\]

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where \( y = \sin \varphi \), whereas the normal elliptic integral of the second kind is

\[
\int_0^y \sqrt{1 - k^2 t^2} dt = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta} d\theta \equiv E(\varphi, k).
\]

The number \( k \) is called the modulus and belongs to the interval \((0, 1)\). The number \( k' = \sqrt{1 - k^2} \) is called the complementary modulus. The parameter \( \varphi \) is called the argument of the normal elliptic integrals. It is usually understood that \( 0 \leq y \leq 1 \) or \( 0 \leq \varphi \leq \frac{\pi}{2} \). For \( y = 1 \), the integrals above are said to be complete. In this case, one writes

\[
\int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \equiv F\left(\frac{\pi}{2}, k\right) \equiv K(k) \equiv K
\]

and

\[
\int_0^1 \frac{1 - k'^2 t^2}{1 - t^2} dt = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta \equiv E\left(\frac{\pi}{2}, k\right) \equiv E(k) \equiv E.
\]

Clearly, we have \( K(0) = E(0) = \frac{\pi}{2} \), while \( E(1) = 1 \) and \( K(1) = +\infty \). For \( k \in (0, 1) \), \( \frac{dK}{dk} > 0 \), \( \frac{dK}{dk} > 0 \), \( \frac{dK}{dk} < 0 \), \( \frac{dE}{dk} < 0 \), and \( E(k) < K(k) \). Moreover, \( E(k) + K(k) \) and \( E(k)K(k) \) are strictly increasing functions for every \( k \in (0, 1) \).

Next, we have some derivatives of the complete elliptical integrals \( K \) and \( E \) used in this work,

\[
\frac{dK}{dk} = \frac{E - k^2 K}{kk'^2}, \quad \frac{dE}{dk} = \frac{E - K}{k}.
\]

The Jacobian elliptic functions are usually defined as follows. It considers the elliptic integral

\[
u(y; k) \equiv u = \int_0^{y_1} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \equiv F(\varphi, k),
\]

which is a strictly increasing function of the variable \( y_1 \). Its inverse function is written \( y_1 = \sin \varphi \equiv sn(u; k) \), or briefly \( y_1 = sn \) when it is not necessary to emphasize the modulus \( k \). The other two basic elliptic functions, the cnoidal and dnoidal functions, are defined in terms of \( sn \) by

\[
\begin{align*}
\text{cn}(u; k) &= \sqrt{1 - y_1^2} = \sqrt{1 - \text{sn}^2(u; k)}, \\
\text{dn}(u; k) &= \sqrt{1 - k^2 y_1^2} = \sqrt{1 - \text{sn}^2(u; k)}.
\end{align*}
\]

Note that these functions are normalized by the requirements \( \text{sn}(0; k) = 0, \text{cn}(0; k) = 1, \) and \( \text{dn}(0; k) = 1 \). The functions \( \text{cn}(\cdot; k) \) and \( \text{dn}(\cdot; k) \) are even functions. These functions are periodic with \( \text{sn}(u + 4K(k); k) = \text{sn}(u; k), \text{cn}(u + 4\sqrt{K(k)}; k) = \text{cn}(u; k), \text{dn}(u + 2K(k); k) = \text{dn}(u; k) \). Moreover, we have the relations \( \text{sn}^2 u + \text{cn}^2 u = 1, k^2 \text{sn}^2 u + \text{dn}^2 u = 1, k^2 \text{sn}^2 u + \text{cn}^2 u = \text{dn}^2 u, \text{sn}(u + 2K(k)) = -\text{sn}(u; k), \text{cn}(u + 4K(k)) = -\text{cn}(u; k) \). We also have the following explicit values: \( \text{sn}(0) = 0, \text{cn}(0) = 1, \text{sn}(K) = 0, \text{cn}(K) = 0 \) and the asymptotic behaviors \( \text{sn}(u; 0) = \sin u, \text{cn}(u; 0) = \cos u, \text{sn}(u; 1) = \text{tanh} u, \text{cn}(u; 1) = \text{sech} u \). Finally, the formulas

\[
\begin{align*}
\frac{\partial}{\partial u} \text{sn} u &= \text{cn} u \text{dn} u, \\
\frac{\partial}{\partial u} \text{cn} u &= -\text{sn} u \text{dn} u, \\
\frac{\partial}{\partial u} \text{dn} u &= -k^2 \text{cn} u \text{sn} u
\end{align*}
\]

are straightforwardly deduced from the foregoing material.
7.2. A nonlocal Poincaré–Wirtinger inequality. The following inequality was used in Proposition 5.1. Suppose \( f \in H^2 \) such that \( \int_{-L}^L f(x)dx = 0 \); then for \( D = \mathcal{H}_x \)

\[
\int_{-L}^L [D^2 f]^2 dx \geq \frac{\pi}{L} \int_{-L}^L f^2 dx.
\]

**Proof.** Let \( f \) be in \( \mathcal{P} \), and consider the Fourier expansion of \( f \) given by

\[
f(x) = \sum_{n=-\infty}^{+\infty} a_n e^{in\frac{\pi}{L}x} = \sum_{n=-\infty}^{+\infty} \hat{f}(n) e^{in\frac{\pi}{L}x},
\]

where \( \hat{f}(n) = \frac{1}{2\pi} \int_{-L}^L f(x)e^{-in\frac{\pi}{L}x} dx \), with \( \hat{f}(0) = 0 \). Then, \( \mathcal{H}_x f(x) = \frac{\pi}{L} \sum_{n=-\infty}^{+\infty} a_n |n| e^{in\frac{\pi}{L}x} \). From the Parseval theorem we obtain

\[
\int_{-L}^L f\mathcal{H}_x f dx = 2L \sum_{n=-\infty}^{+\infty} \frac{\pi}{L} |a_n|^2 |n| = 2L \sum_{n \neq 0} \frac{\pi}{L} |a_n|^2 |n|
\]

and

\[
\int_{-L}^L f^2 dx = 2L \sum_{n \neq 0} |a_n|^2.
\]

Therefore,

\[
\int_{-L}^L f\mathcal{H}_x f dx = 2L \sum_{n \neq 0} \frac{\pi}{L} |a_n|^2 |n| \geq \frac{\pi}{L} \int_{-L}^L f^2 dx.
\]

So, using density arguments, we can show that the inequality above occurs for \( f \in H^2 \). This completes the proof of the inequality. \( \square \)

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